3. Optical Resonator

3.1 The Plane Mirror Resonator
3.2 The Spherical Mirror Resonator
3.3 Gaussian modes and resonance frequencies
3.4 The Unstable Resonator
Introduction

- The optical resonator (OR) is the optical counterpart of an electronic resonant circuit: it confines and stores light at certain frequencies.
- Most important application: OR as a container within which laser light is generated.
- LASER=OR containing a light amplifying medium
- OR determines frequency and spatial distribution of the laser beam
Introduction

From Fundamentals of Photonics, Saleh and Teich, Wiley, chap 10, p.366
Introduction

- Mirror resonators: 2 or 3 mirrors, 2D or 3D cavities
- Dielectric Resonators: use TIR instead of mirrors:
  - Fiber rings and integrated optic rings
  - Microdisks, microspheres, etc (Whispering Gallery modes)
  - Micropillars
  - Photonic Crystals
- Currently: Nanolasers with quantum confinement of carriers (e.g. electrons) or photons
Introduction

- Two key parameters:
  - Modal volume $V$: volume occupied by confined optical mode
  - Quality factor $Q$: proportional to storage time in units of optical period
- $V$ and $Q$ represent the degrees of spatial and temporal confinements, respectively
- Large $Q$ means low-loss resonator
3.1- Plane Mirror Resonator

- Fabry-Perot interferometer: pair of plane mirrors separated by distance d.

Monochromatic plane: \( u(\vec{r}) = \text{Re}[U(\vec{r})e^{2\pi i \nu t}] \)

Satisfies Helmholtz equation:

\[ \nabla^2 U + k^2 U = 0 \quad \text{with} \quad k = 2\pi \frac{\nu}{c} \]
3.1- Plane Mirror Resonator

- Standing wave solution is obtained for the boundary conditions:

\[
U(\vec{r}) = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad U(\vec{r}) = 0 \quad \text{at} \quad z = d
\]

\[
U(\vec{r}) = A \sin k z \quad \text{with} \quad kd = q \pi \quad \Rightarrow \quad k = q \frac{\pi}{d}, \quad q = 1, 2, 3, \ldots
\]

- **q** is the mode number, mode frequencies are:

\[
\nu_q = q \frac{c}{2d}
\]

- Arbitrary wave = superposition of modes

\[
U(r) = \sum_q A_q \sin k_q z
\]

- Constant frequency difference between adjacent modes (free spectral range): \( \nu_F = \frac{c}{2d} \)
3.1- Plane Mirror Resonator

- The resonance wavelengths in the optical medium are: \( \lambda_q = \frac{c}{\nu_q} \Rightarrow 2d = q\lambda_q \)

- Examples:
  - \( d = 30 \text{ cm}, n = 1 \text{ (air)}, \) free spectral range = 500 MHz
  - \( d = 3 \text{ microns}, n = 1 \text{ (air)}, 50 \text{ THz} \) (7 modes in visible range: \( q = 8, \ldots, 14, \lambda_q = 750, \ldots, 429 \text{ nm} \))

- Free spectral range can be adjusted by placing resonators in series
3.1- Plane Mirror Resonator

- Calculation of light intensity in the resonator:
  - Summation of multiply reflected amplitudes
  - Phase shift after one round trip of propagation (2d) is \( \varphi = \frac{2\pi}{\lambda} 2d = 2kd \)
  - Wave reproduces itself after a round trip, thus: \( \frac{2\pi}{\lambda} 2d = 2kd = 2q\pi, q = 1, 2, 3... \)

\[
U(r, t) = U_0 e^{i(kr - 2\pi vt)}, \quad k.r = \frac{2\pi}{\lambda} 2d \text{ for } r = 2d
\]
3.1- Plane Mirror Resonator

Calculation of light intensity in the resonator:

- Summation of multiply reflected amplitudes for perfect “non lossy” resonator:

\[ U_1 = U_0 e^{-i \frac{2\pi}{\lambda} 2d}, U_2 = U_1 e^{-i \frac{2\pi}{\lambda} 2d} = U_0 e^{-i \frac{2\pi}{\lambda} 4d}, \text{etc...} \]

Total amplitude:

\[ U = U_0 + U_1 + U_2 + ... \]

\[ U_{Total} = U = U_0 \sum_{j=0}^{\infty} e^{i \frac{2\pi}{\lambda} 2jd} \]
3.1-Plane Mirror Resonator

- If resonator has losses, amplitude reduction upon reflection is taken into account (r reflection coefficient):

Total amplitude: \( U = U_0 + rU_1 + r^2U_2 + \ldots \)

\( r = \) complex reflection coefficient (overall amplitude attenuation)

Intensity: \( I = |U|^2 \)

\[
I = \frac{I_0}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2\left(\frac{\varphi}{2}\right)} = \text{transmitted intensity}
\]

\( I_0 = |U_0|^2 = \text{incident intensity} \)

\( R = |r|^2, \) reflectivity of lossy mirror (or overall losses over round trip)

\[
F = \frac{\sqrt{|r|}}{1 - |r|} \quad \text{Finesse of resonator} = \frac{\text{Intermode spacing}}{\delta \nu} = \frac{\nu_F}{\delta \nu}
\]

\[
U_{\text{Total}} = U = U_0 \sum_{j=0}^{N} r^j e^{i(\frac{2\pi}{\lambda})2jd}
\]
3.1- Plane Mirror Resonator

Spectral width depends strongly on finesse (F). Cf. F=100 vs 10…

Spectral Response of the lossy resonator:

\[
\frac{I}{I_0} = \frac{1}{1 + \left( \frac{2F}{\pi} \right)^2 \sin^2 \left( \pi \frac{\nu}{\nu_F} \right)}
\]

\[\nu_F = \frac{c}{2d}, I_0 \equiv \text{Incident intensity}\]

\[\frac{1}{\nu_F} = \text{round-trip time}\]

Spectral width of a mode \( \approx \frac{\nu_F}{F} \)
3.1- Plane Mirror Resonator

- The two principal sources of loss in the optical resonator are
  - Absorption and scattering in the medium between the mirror (see laser amplifier):
    Round trip power attenuation: \( \exp(-2\alpha_s d) \)

  \( \alpha_s \): linear absorption coefficient of the medium

  - Losses arising from imperfect reflection at the mirrors (necessary transmission + finite size effects):

  Mirrors of reflectance: \( R_1 \) and \( R_2 \)

  Overall round trip loss of intensity: \( R_1 R_2 \exp(-2\alpha_s d) \equiv r^2 \)

Overall distributed-loss written as:

\[
\exp(-2\alpha_r d) = R_1 R_2 \exp(-2\alpha_s d)
\]

\[
\alpha_r = \alpha_s + \frac{1}{2d} \log \frac{1}{R_1 R_2}
\]

Ultimately (after maths): \( F \approx \frac{\pi}{\alpha_r d} \)

if \( \alpha_r d << 1 \) (small losses)
The resonance linewidth is inversely proportional to the loss factor ($\alpha_r d$)

\[
F = \frac{\nu_F}{\delta\nu} \rightarrow \delta\nu \approx \frac{c}{\frac{2d}{\pi}} = \frac{c\alpha_r}{2\pi \alpha_r d}
\]

$\alpha_r$ is the loss per unit length, $c\alpha_r$ is the loss per unit time.

The resonator lifetime or photon lifetime in cavity is:

\[
\tau_p = \frac{1}{c\alpha_r}, \text{ thus } \delta\nu = \frac{1}{2\pi \tau_p}
\]
3.1 Plane Mirror Resonator

- The Quality factor $Q$ can be used to characterise the losses:

$$Q = 2\pi \frac{\text{Stored energy}}{\text{Energy loss per cycle}}$$

In the case of an optical resonator (laser), one can show that:

$$Q = 2\pi \nu_0 \tau_p$$

$$Q = \frac{\nu_0}{\nu_F} F, \quad \nu_0 = \text{frequency of one of the modes}$$

- Since the resonator frequencies are much larger than the mode spacing, then $Q \gg F$
3.1-Plane Mirror Resonator

- What are the requirements for a laser:
  - Assume 3-D resonator (3 pairs of parallel mirrors, closed resonator), equivalent to black-body cavity.
  - Number and frequency of modes is given by the particle in the box model (photons):
    \[
    \frac{dN}{V} = \frac{8\pi
    \nu^2}{c^3} \, d\nu
    \]
    \[
    V = 1\, cm^3, \nu = 3 \times 10^{14} \, Hz, \, d\nu = 3 \times 10^{10} \, Hz
    \]
    \[
    dN = 2 \times 10^9 \, \text{modes}
    \]
3.1-Plane Mirror Resonator

- All the modes would have comparable Q in the 3D resonator
  - To be avoided in a laser as it would cause all the atoms to emit power into a large number of modes (would differ in their frequency and spatial characteristics)

- Large, open resonators consisting of opposite flat/curved reflectors must be used:
  - Energy of the vast majority of modes lost after a single pass
  - Surviving modes are near the axis
3.2 Spherical Mirror Resonator

- Ray confinement:
  - Concave $R<0$,
  - Convex $R>0$,
  - Only meridional (lie in a plane passing through the optical axis) and paraxial rays are considered

- Geometric optics is sufficient to find the condition for the existence of the confined modes

$$g_1 = 1 + \frac{d}{R_1}, \quad g_2 = 1 + \frac{d}{R_2}$$

3.2 Spherical Mirror Resonator

- Condition for the existence of confined modes:
  - Outside this domain the resonator is said to be unstable

\[ 0 \leq g_1 g_2 \leq 1 \]

- For same radii, stability condition becomes:

\[ R_1 = R_2 \Rightarrow g_1 = g_2 = g \]
\[ -1 \leq g \leq +1 \Rightarrow 0 \leq \frac{d}{(-R)} \leq 2 \]

- Three resonators of practical interest: confocal concentric, confocal/planar
3.2-Spherical Mirror Resonator

Stability condition:

\[ R_1 = R_2 \Rightarrow g_1 = g_2 = g \]

\[-1 \leq g \leq +1 \Rightarrow 0 \leq \frac{d}{(-R)} \leq 2\]

- a. Planar \((R_1 = R_2 = \infty)\)
- b. Symmetrical confocal \((R_1 = R_2 = -d)\)
- c. Symmetrical concentric \((R_1 = R_2 = -d/2)\)
- d. Confocal/planar \((R_1 = -d, R_2 = \infty)\)
- e. Concave/convex \((R_1 < 0, R_2 > 0)\)
3.3 Gaussian Modes and resonance frequencies

- Gaussian beams are stable modes of the spherical mirror resonator: wavefronts and phase match exactly the boundary conditions imposed by spherical mirror resonator (Helmholtz paraxial equation).

- Gaussian beam retraces incident beam if the radius of the wavefronts is exactly the same as the mirror radius.

- Phase of Gaussian beam:
  \[ \varphi(R, z) = k z - \xi(z) + \frac{k \rho^2}{2 R(z)} \text{ with } \rho^2 = x^2 + y^2 \]

  On-axis: \[ \varphi(0, z) = k z - \xi(z) \]

  \[ \xi(z) = \arctan\left(\frac{z}{z_0}\right) : \text{phase retardation with respect to plane wave} \]
3.3 Gaussian Modes and resonance frequencies

- See chapter 10 for details (symmetrical resonator):

\[ R_1 = R_2 = -|R| \]
\[ z_1 = -d/2, \quad z_2 = d/2 \]
\[ R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right]^{1/2} \]
\[ W(z) = W_0 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]^{1/2} \]

\[ z_0 = \frac{d}{2} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_0^2 = \frac{\lambda d}{2\pi} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_1^2 = W_2^2 = \frac{\lambda d}{2\pi} \left[ \frac{d}{|R|} \left( 2 - \frac{d}{|R|} \right) \right]^{1/2} \]
3.3 Gaussian Modes and resonance frequencies

\[ z_2 = z_1 + d \quad R(z) = z + \frac{z_0^2}{z} \]

\[ R_1 = z_1 + \frac{z_0^2}{z_1}, \quad -R_2 = z_2 + \frac{z_0^2}{z_2} \]

\[ z_1 = \frac{-d(R_2 + d)}{R_2 + R_1 + 2d} \]

\[ z_0^2 = \frac{-d(R_1 + d)(R_2 + d)(R_1 + R_2 + d)}{(R_2 + R_1 + 2d)^2} \]
3.3 Gaussian Modes and resonance frequencies

- See chapter 10 for details (symmetrical resonator):

\[ R_1 = R_2 = -|R| \]
\[ z_1 = -d/2, \ z_2 = d/2 \]
\[ R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] \]
\[ W(z) = W_0 \left[ 1 + \left( \frac{z}{24z_0} \right)^2 \right]^{1/2} \]

\[ z_0 = \frac{d}{2} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_0^2 = \frac{\lambda d}{2\pi} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_1^2 = W_2^2 = \frac{\lambda d}{\pi} \frac{1}{\left\{ \left( \frac{d}{|R|} \right) \left[ 2 - \left( \frac{d}{|R|} \right) \right] \right\}^{1/2}} \]
3.3 Gaussian modes and resonance frequencies

- Resonance frequencies can be calculated from the resonance condition (round trip phase change is exactly $2\pi$):

At mirrors location:

$\varphi(0, z_1) = kz_1 - \xi(z_1)$ and $\varphi(0, z_2) = kz_2 - \xi(z_2)$

Phase change from $z_1$ to $z_2$:

$\Delta \varphi = \varphi(0, z_2) - \varphi(0, z_1) = k(z_2 - z_1) - [\xi(z_2) - \xi(z_1)] = kd - \Delta \xi$

For one round trip + phase matching condition:

$\Delta \varphi = 2(kd - \Delta \xi) = 2q\pi\ (q = \pm 1, \pm 2, ...)$

$v_q = qv_F + \frac{\Delta \xi}{\pi}v_F, \quad$ frequency spacing: $\left(v_F = \frac{c}{2d}\right)$
3.3 Gaussian modes and resonance frequencies

- All the Hermite-Gaussian beams of order \((l,m)\) are also good solutions.
- All \((l,m)\) modes have same wavefronts as \((0,0)\) but different amplitudes: Conditions for wavefront matching are identical.
- The entire family of \(A_{l,m} G_l G_m\) are also modes of the spherical mirror resonator
- The resonance frequencies depend on \((l,m)\)
Gaussian Beams - Modes

Intensity distribution of Hermite-Gaussian modes:

\[ I_{l,m}(x,y,z) = |A_{l,m}|^2 \left[ \frac{W_0}{W(z)} \right]^2 G_l^2 \left( \frac{\sqrt{2}x}{W(z)} \right) G_m^2 \left( \frac{\sqrt{2}y}{W(z)} \right) \]

TEM\(_{lm}\) modes: \(G_l, G_m\) Hermite-Gaussian function of order \(l, m\)

\(A_{l,m} = \text{constant} \ (l, m)\)

TEM\(_{00}\) = Gaussian Beam
3.3 Gaussian modes and resonance frequencies

- Phase matching conditions provide resonance frequencies:

  Phase of the axial modes:
  \[ \varphi(0,z) = k z - (l + m + 1) \xi(z) \]

  After a round trip + phase matching condition
  \[ 2kd - 2(l + m + 1) \Delta \xi = 2q\pi \quad (q = \pm 1, \pm 2, \ldots) \]

  Resonance frequencies:
  \[ \nu_q = q\nu_F + (l + m + 1) \frac{\Delta \xi}{\pi} \nu_F \]

- Modes of different \( q \) but same \( (l,m) \) are called **longitudinal (axial)** modes

- Modes with different \( (l,m) \) represent different **transverse** modes
3.4 Unstable Resonator

- Close to regions of ‘unconfinement’, beam size increases
- Light losses due to missing the mirror become important (diffraction losses).
- For high power applications, large volume modes and diffraction losses are desirable
- High diffraction losses are good for a high gain situation (see later).
- Output beam has large aperture: optics are simplified
- Losses depend only on mirrors radii of curvature and separation distance.
3.4 Unstable Resonator

- Spherical wave picture of the mode in an unstable resonator.
- Points $P_1$ and $P_2$ are the virtual centres of the spherical waves.