Chapter 2
The Gaussian Beam

2.1 The Gaussian Beam

2.2 The TEM_{00} mode

2.3 Beam quality: M^2 factor

2.4 Transmission of Gaussian Beam through thin lenses

2.1-The Gaussian Beam

- The Gaussian Beam is an important solution of the Helmholtz (Maxwell) paraxial wave equation(s).
- The Gaussian Beam solutions are the modes of the spherical mirror optical resonator (See III. Optical resonator).
- The optics of a laser beam is essentially that of the Gaussian beam

https://en.wikipedia.org/wiki/Helmholtz_equation

2.1-The Gaussian Beam

$U(\vec{r}) = A(\vec{r})e^{-ikz}$

$A(\vec{r})$ variation with position is very small over a distance of one λ.

It is still approximately planar.

- The Helmholtz Equation in the Paraxial Approximation becomes:

$$\nabla_T^2 A - i2k\frac{\partial A}{\partial z} = 0$$

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \text{Transverse Laplace operator}$$

2.1-The Gaussian Beam

- The Gaussian beam solution of Maxwell's equations for the electric vector E is given by:

$$\vec{E}(\vec{r}) = E_0\left(-\hat{x} + \frac{x}{z + iz_0}\hat{z}\right)U(\vec{r})$$

- $\hat{x}$ and $\hat{z}$ are units vectors in the 0x and 0z directions respectively and U(r) is the complex amplitude of the scalar Gaussian beam.

2.1-The Gaussian Beam

• ***U(r)*** is written in the form [magnitude X exp(-iphase)]

$$U(\vec{r}) = A(\vec{r})e^{-ikz}$$

$$A(\vec{r}) = A_0 \frac{W_0}{W(z)} exp\left(-\frac{\rho^2}{W^2(z)}\right) exp\left(-ik\frac{\rho^2}{2R(z)} + i\zeta(z)\right)$$

The beam parameters are:

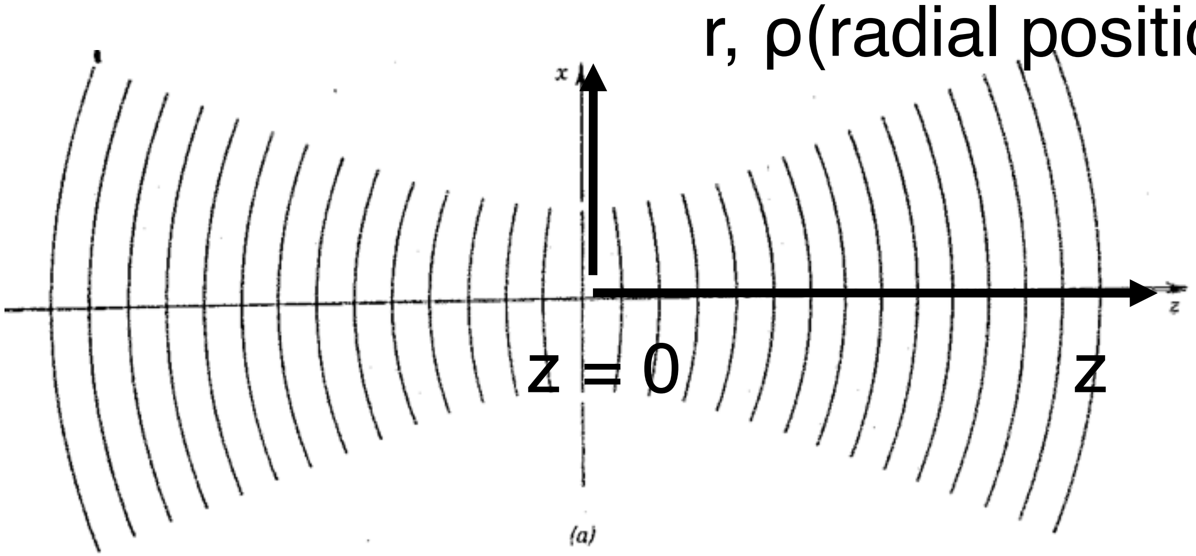
$W(z)$: Beam width = radius (!)

W_0 : Beam waist

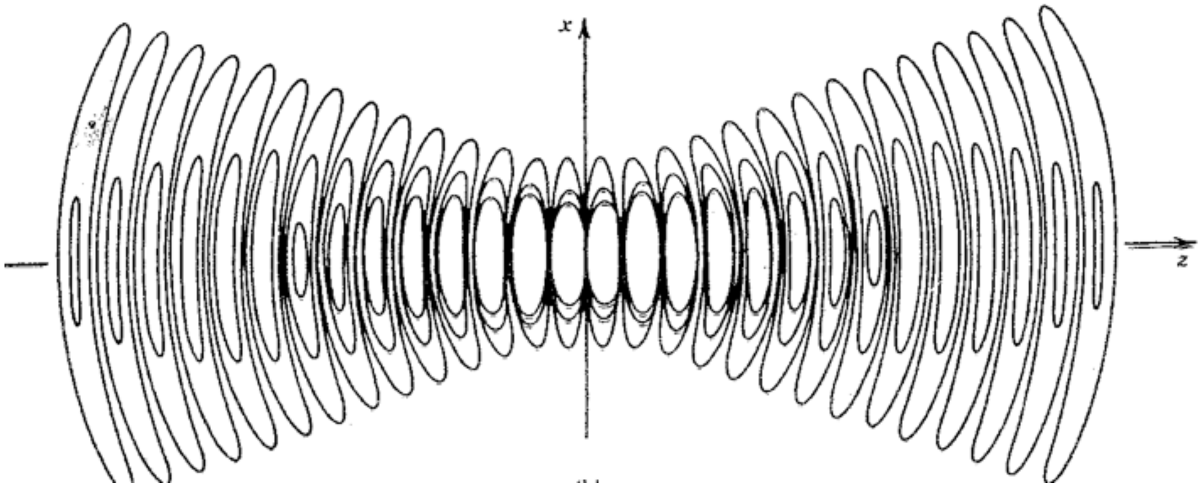
$R(z)$: Radius of curvature of wavefronts

$\zeta(z)$: Phase factor

2.1-The Gaussian Beam

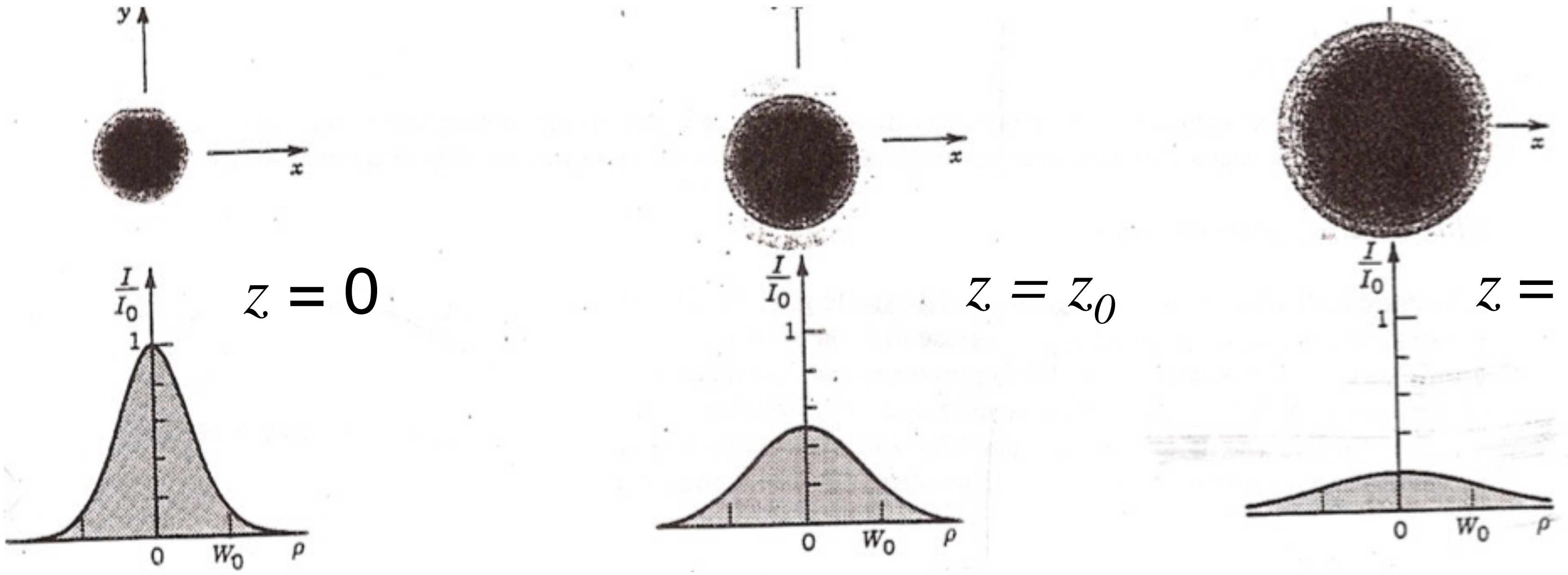


- Wavefronts of scalar Gaussian Beam (cylindrical coordinates)



- Electric field lines in (x-z) plane

2.1-The Gaussian Beam



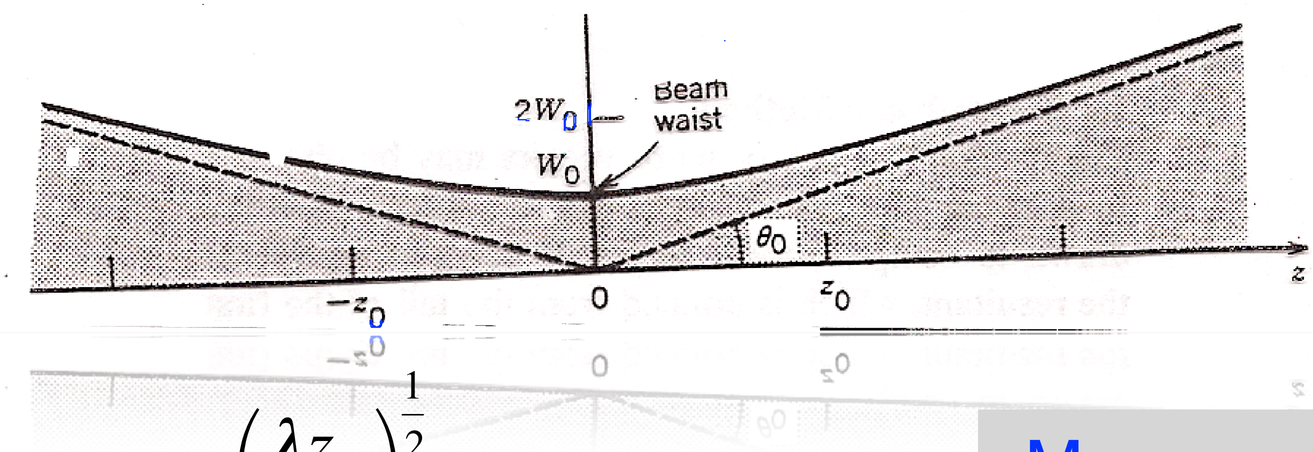
$$I(\vec{r}) = |U(\vec{r})|^2$$

$$I(\vec{r}) = I(\rho, z)$$

$$= I_0 \left(\frac{W_0}{W(z)} \right)^2 exp\left(-\frac{2\rho^2}{W^2(z)}\right)$$

- At $z = z_0$, the on-axis intensity is halved
- z_0 is called the Rayleigh range

2.1-The Gaussian Beam



$$W(z) = W_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{\frac{1}{2}}$$

$$W_0 = \left(\frac{\lambda z_0}{\pi}\right)^{\frac{1}{2}}$$: Beam Waist

z_0 : the Rayleigh range

$$W(z) \approx \frac{W_0}{z_0} z = \theta_0 z$$

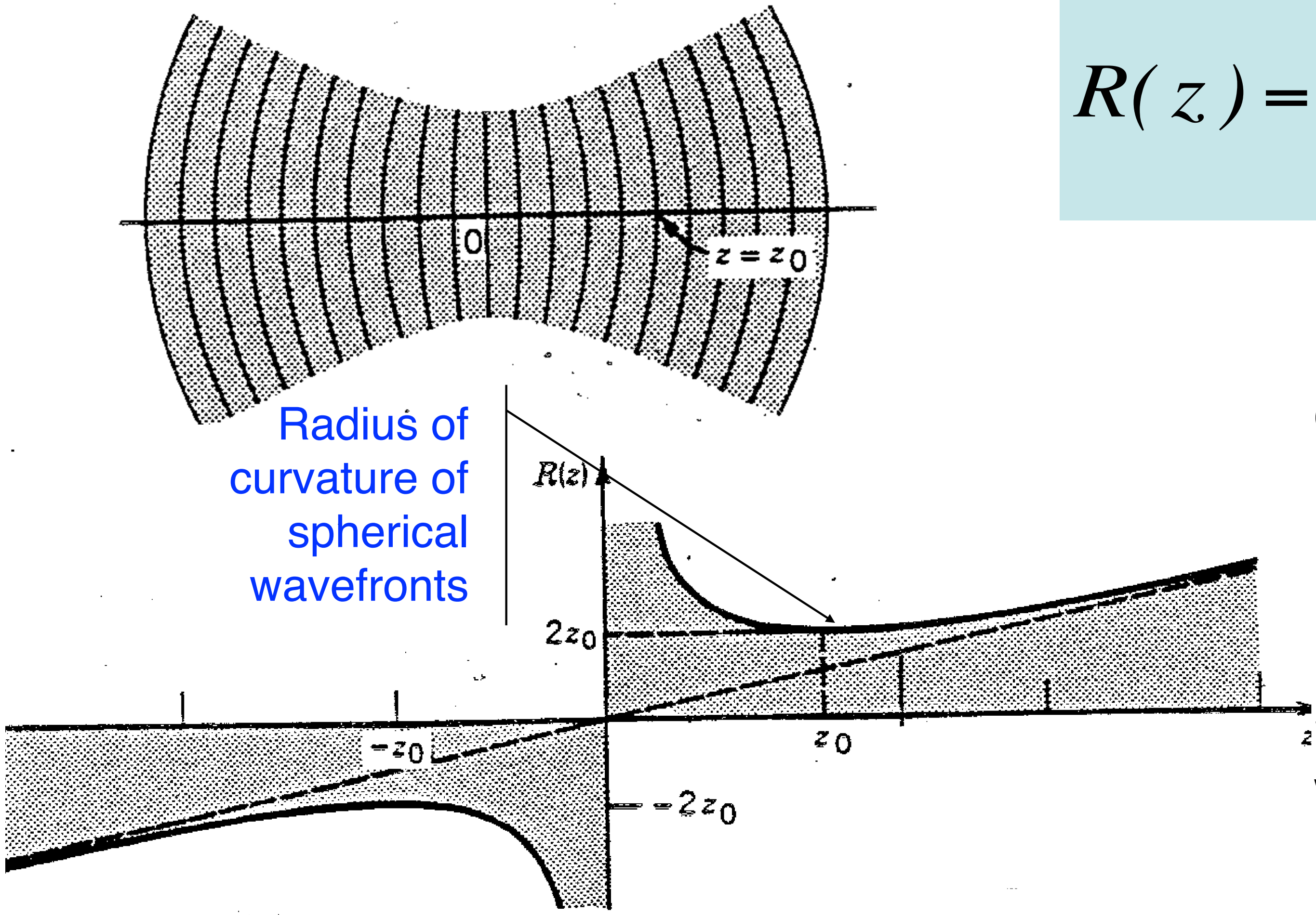
$$\theta_0 = \frac{\lambda}{\pi W_0}$$: Beam Divergence

- Measurement of the beam waist provides z_0.
- e.g: 266 nm (quadrupled Nd-YAG laser), W_0 = 2.5 mm:
  - z_0 = 75 m (depth of focus)
  - θ_0 = 0.01 mrad

2.1-The Gaussian Beam

- Wavefronts and their radius of curvature

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

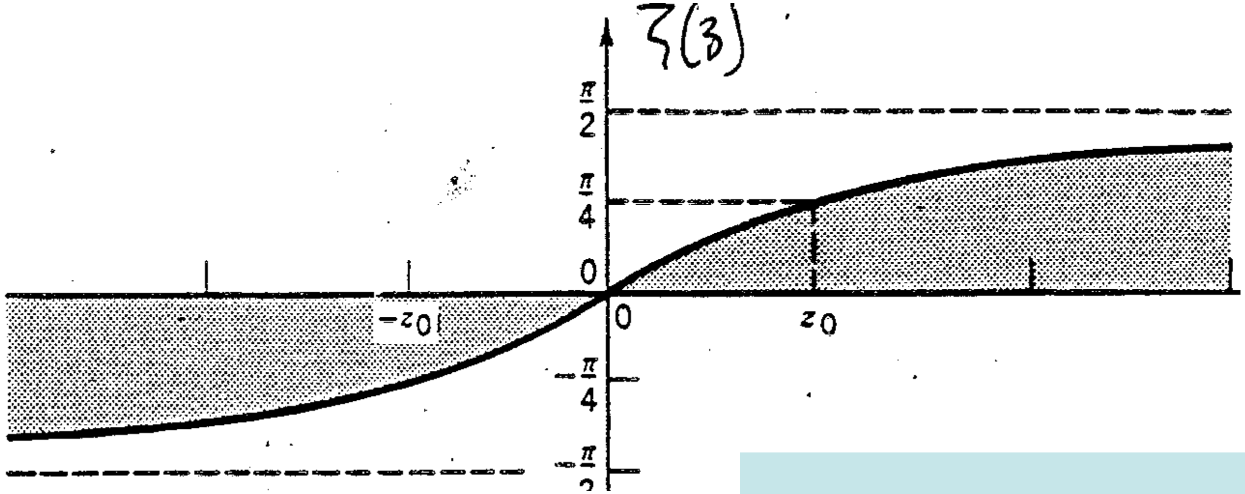


•Wavefronts curvature is minimum at z_0.

•Same as spherical wavefronts at large z $R(z) \approx z$

2.1-The Gaussian Beam

- The phase of the Gaussian Beam:



$$\varphi(R,z) = kz - \zeta(z) + \frac{k\rho^2}{2R(z)}$$

$$\varphi(0,z) = kz - \zeta(z)\text{: on-axis}$$

$$\zeta(z) = arctan\left(\frac{z}{z_0}\right)\text{: phase retardation}$$

with respect to plane wave

2.1-The Gaussian Beam

- Most general solutions are Hermite-Gaussian functions (higher order Gaussian beams).

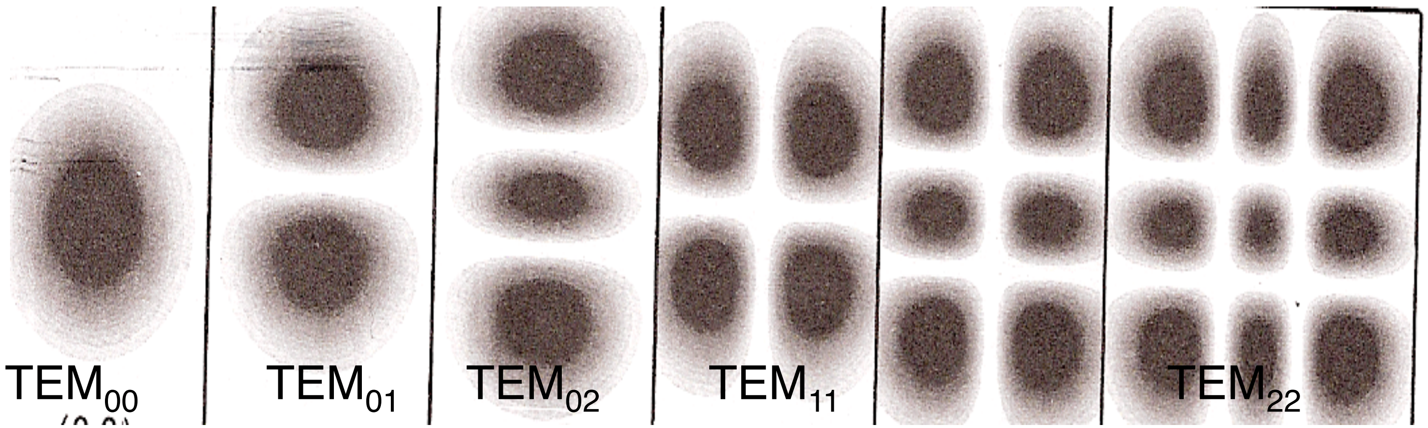
Intensity distribution of Hermite-Gaussian modes:

$$I_{l,m}(x,y,z) = \left|A_{l,m}\right|^2 \left[\frac{W_0}{W(z)}\right]^2 G_l^2\left(\frac{\sqrt{2}x}{W(z)}\right) G_m^2\left(\frac{\sqrt{2}y}{W(z)}\right)$$

TEM_{lm} modes: G_l, G_m Hermite-Gaussian function of order l, m

$A_{l,m}$ = constant (l, m)

TEM_{00} = Gaussian Beam



2.1-The Gaussian Beam

Hermite Gaussian Functions - Cartesian Coordinates

https://www.rp-photonics.com/hermite_gaussian_modes.html

$$E_{nm}(x,y,z) = E_0 \frac{w_0}{w(z)}$$

$$\cdot H_n\left(\sqrt{2}\frac{x}{w(z)}\right)\exp\left(-\frac{x^2}{w(z)^2}\right)\cdot H_m\left(\sqrt{2}\frac{y}{w(z)}\right)\exp\left(-\frac{y^2}{w(z)^2}\right)$$

$$\cdot \exp\left(-i\left[kz-(1+n+m)\arctan\frac{z}{z_R}+\frac{k\left(x^2+y^2\right)}{2R(z)}\right]\right)$$

- The first eleven physicists' Hermite polynomials are:

$$\begin{aligned}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= 4x^2 - 2, \\
H_3(x) &= 8x^3 - 12x, \\
H_4(x) &= 16x^4 - 48x^2 + 12, \\
H_5(x) &= 32x^5 - 160x^3 + 120x, \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120, \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x, \\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680, \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x, \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.
\end{aligned}$$

https://en.wikipedia.org/wiki/Hermite_polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1.$$

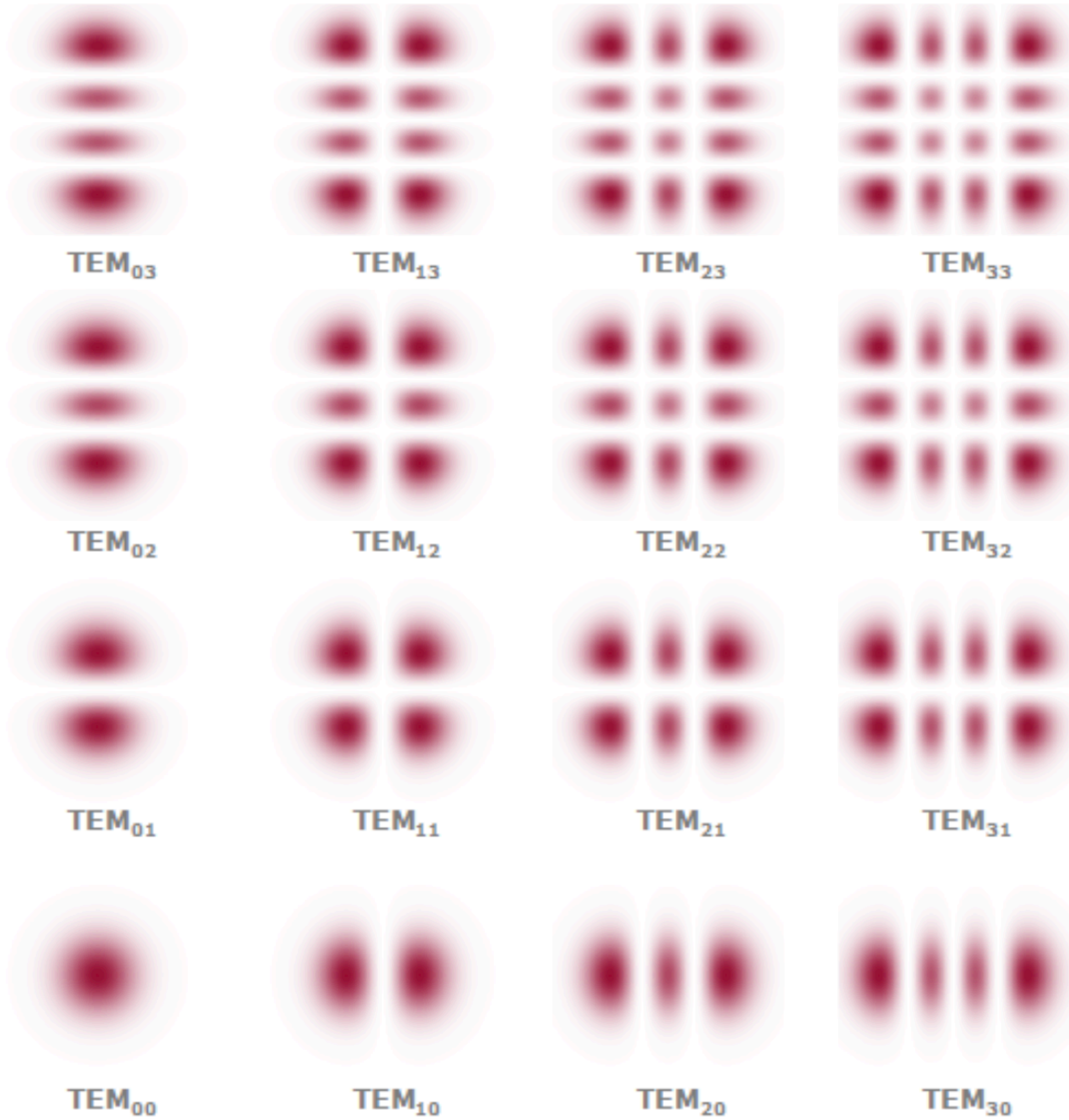


Figure 1: Intensity profiles of the lowest-order Hermite–Gaussian modes, starting with TEM_{00} (lower left-hand side) and going up to TEM_{33} (upper right-hand side).

https://www.rp-photonics.com/hermite_gaussian_modes.html

2.2 - TEM_{00} mode

- Normal or Gaussian (non-normalised) function:

$$G(x) = e^{-\frac{x^2}{2\sigma^2}}$$

σ rms (root mean square) width

- Generally:

$$g(x) = e^{-\alpha x^2}$$

$$G(0) = g(0) = 1$$

$$\lim_{x \to \pm\infty} G(x) \to 0$$

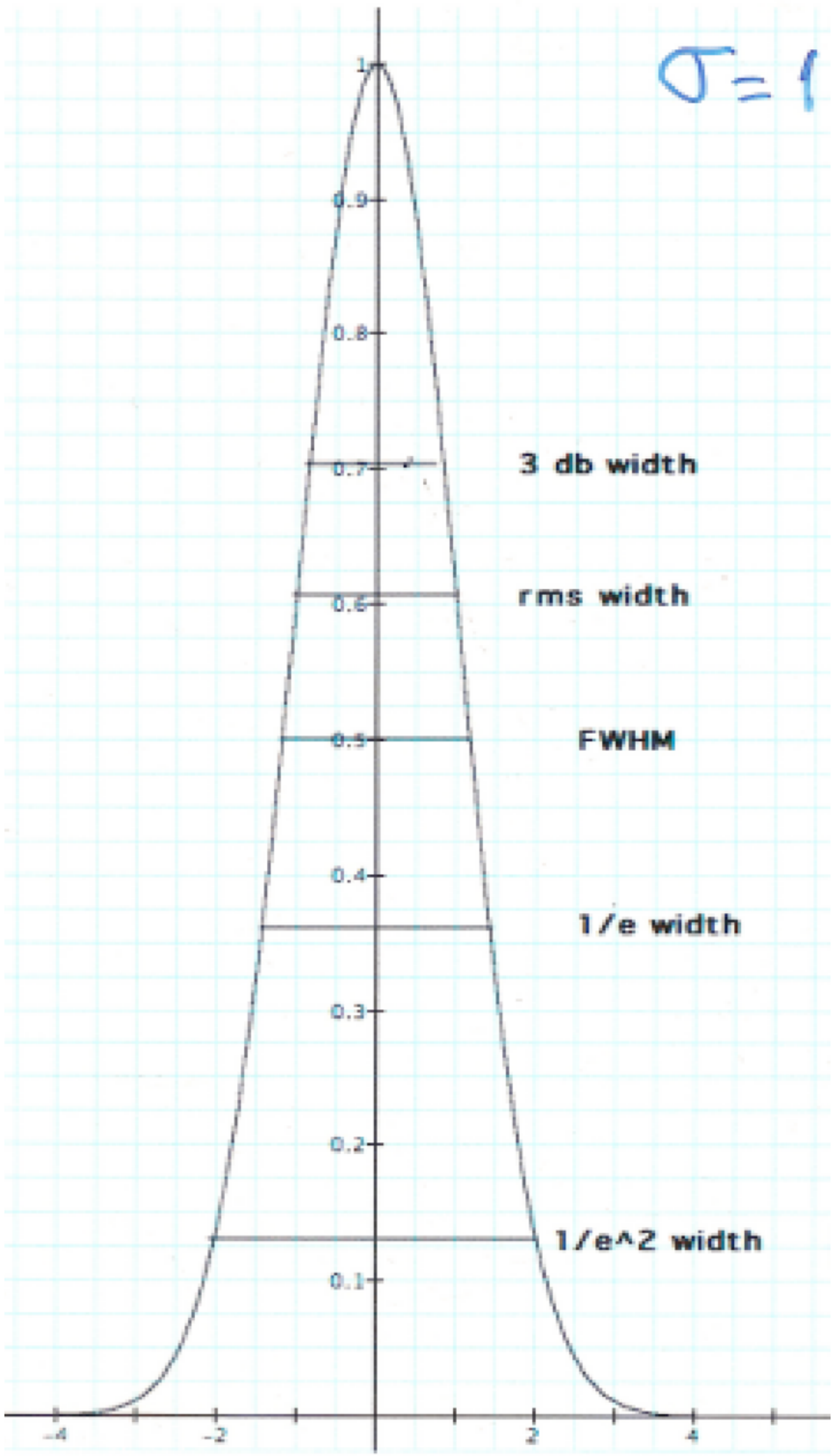
- If G(x) represents light intensity, how does one define the "edge" of the beam?

2.2 - TEM_{00} mode

- Need to truncate G(x) at certain values of x (x_t measured from I_{max})
- $2x_t$ defines the corresponding width

I_{max} × attenuation	Truncation @
$1/\sqrt{2} = 0.707 \times 1$	0.83σ (3 dB)
$1/\sqrt{e} = 0.606 \times 1$	σ (rms width)
$1/2 = 0.5 \times 1$	1.18σ (FHWM)
$1/e = 0.368 \times 1$	1.414σ (1/e)
$1/e^2 = 0.13 \times 1$	2σ ($1/e^2$ width)

(see graph on next page)



2.2 - TEM_{00} mode

- The intensity can be written as a function of P: total optical power carried by the beam. P is measured directly with a power meter.

$$P = \int_0^\infty I(\rho,z)2\pi\rho d\rho = \frac{1}{2} I_0\left(\pi W_0^2\right)$$

$$I(\rho,z) = \frac{2P}{\pi W^2(z)} exp\left(-\frac{2\rho^2}{W^2(z)}\right)$$

- The beam radius is taken as the ($1/e^2$) width: spot size

2.2 - TEM$_{00}$ mode

- For a given pulse duration, it is convenient to use fluence (Jcm^{-2}) -instead of intensity -as a number of laser processes are characterised by their fluence (eg. laser ablation threshold):

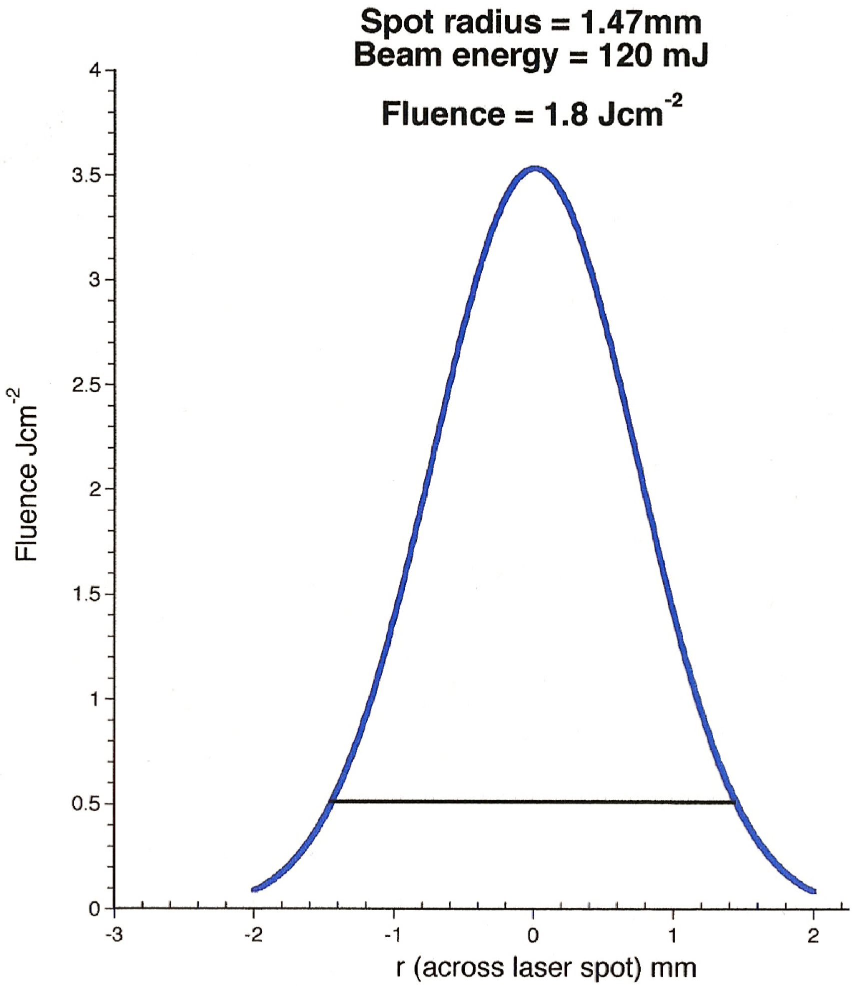
$$F_z(r) = \frac{2E_T}{\pi W_z^2} exp\left[-2\left(\frac{r^2}{W_z^2}\right)\right]$$

E_T = Total energy in laser beam

W_z = $\frac{1}{e^2}$ spot size on target (at z)

$\frac{E_T}{\pi W_z^2}$ ≡ Average fluence

Nd:YAG laser, 6 ns, 120 mJ, 266 nm



2.3 Beam Quality: M^2 factor

- Gaussian beam is an idealisation
- Deviation of optical beam (waist diameter $2W_m$, divergence $2\theta_m$) from Gaussian form (W_0, θ_0) measures optical quality: quantitative measure is M^2 -factor:

$$M^2 = \frac{2W_m 2\theta_m}{2W_0 2\theta_0} = \frac{2W_m 2\theta_m}{4\lambda / \pi}$$

$$M^2 = \frac{\theta_m}{\theta_0}$$ If the two beams have the same beam waist

$M^2 \geq 1$

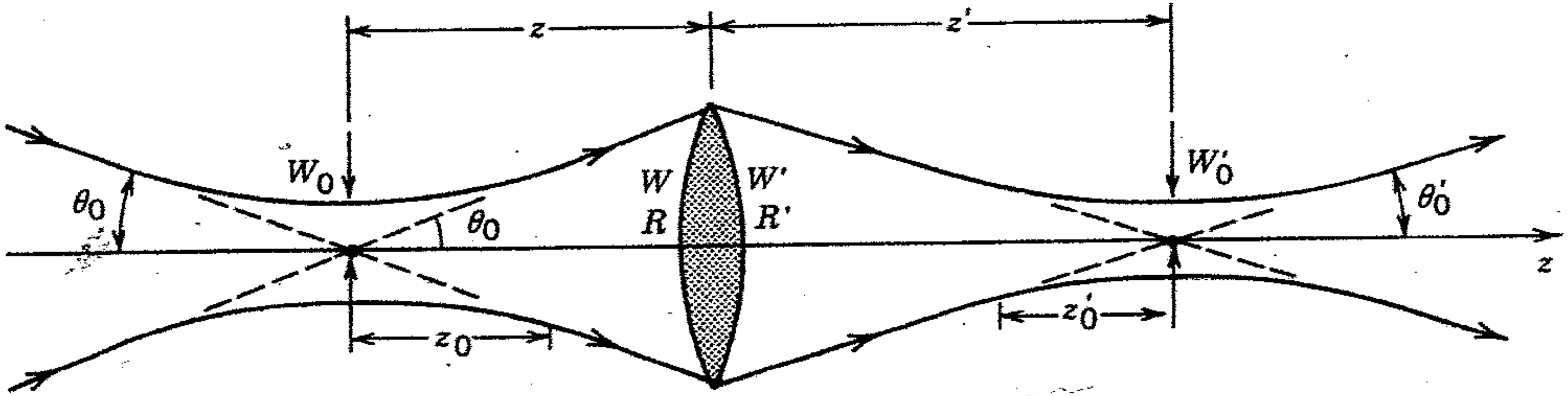
$M^2 \leq 1.1$ (single mode HeNe laser),

$M^2 \geq 3{,}4$ (high power multimode)

2.4 Transmission of Gaussian beams through thin lenses

- Gaussian beam remains Gaussian if paraxial nature of the wave is maintained.
- Beam is reshaped: waist and curvature are altered.
- Beam shaping, beam focusing can be achieved (optical design)

2.4 Transmission of Gaussian beams through thin lenses



Transmission of a Gaussian beam through a thin lens.

- Complex amplitude multiplied by phase factor as it passes through lens
- Wavefront is altered: new curvature, new phase (beam width $W = W'$).

2.4 Transmission of Gaussian beams through thin lenses

- Parameters of the emerging beam:

$$\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$$

$$W_0' = \frac{W}{\sqrt{1+\left(\pi W^2/\lambda R'\right)^2}} \quad \text{(Waist radius)}$$

$$-z' = \frac{R'}{1+\left(\lambda R'/\pi W^2\right)^2} \quad \text{(Beam centre location)}$$

f = focal length of thin lens

Waist radius: $\mathrm{W}_0' = M\mathrm{W}_0$

Waist location: $(z' - f) = M^2 (z - f)$

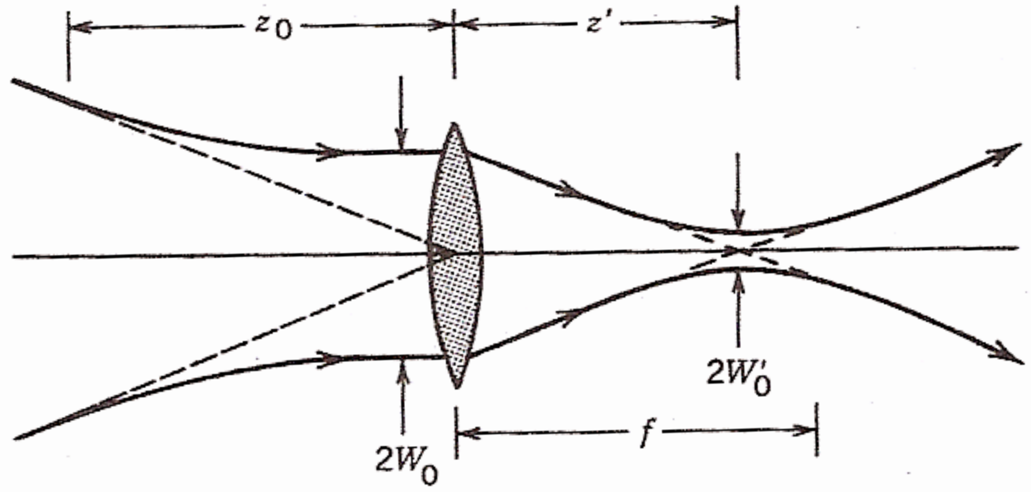
Depth of focus: $2z_0' = M^2 2z_0$

Divergence angle: $2\theta_0' = \dfrac{2\theta_0}{M}$

Magnification: $M = \dfrac{\mathrm{M_r}}{\sqrt{1+\mathrm{r}^2}}$

$$r = \frac{z_0}{z-f} \quad \text{and} \quad \mathrm{M_r} = \left|\frac{f}{z-f}\right|$$

2.4 Transmission of Gaussian beams through thin lenses



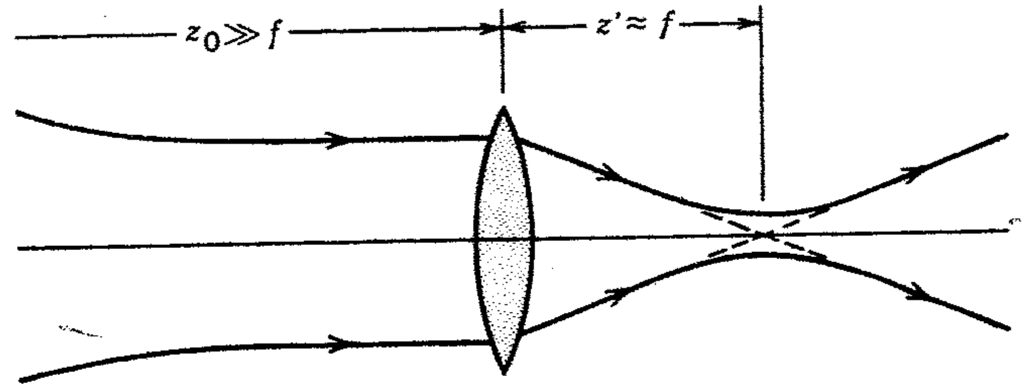
Focusing a beam with a lens at the beam waist.

$$W_0' = \frac{W}{\sqrt{1+\left(z_0/f\right)^2}} \quad \text{(Waist radius)}$$

$$z' = \frac{R'}{1+\left(f/z_0\right)^2} \quad \text{(Beam centre location)}$$

- Beam shaping: use lens or series of lenses to reshape the Gaussian beam
- Lens at beam waist: do $z = 0$ in previous equations

2.4 Transmission of Gaussian beams through thin lenses



Focusing a collimated beam.

$$W_0' \approx \frac{f}{z_0} W_0 = \frac{\lambda}{\pi W_0} f = \theta_0 f \quad \text{(Waist radius)}$$

$$z' \approx f \quad \text{(Beam centre location)}$$

- Depth of focus $z_0 >> f$
- Small spot size is really important in laser scanning, laser printing, CD burning,...
- Need short f, thick beam and short wavelength
- If D (diameter of lens) $= 2W_0$

Focused spot size:

$$2W_0' \approx \frac{4}{\pi} \lambda \frac{f}{D} = \frac{4}{\pi} \lambda F\#$$

$F\# \rightarrow$ F-number of lens