I-1 Wave Optics: definitions

- From experimental evidence: light propagates in the form of waves
- Light wave (vibration) = scalar wave = wavefunction
- This description accounts for a large number of optical phenomena
- Nature of light remains unspecified.
I-1 Wave Optics: definitions

- Wave travels in a homogeneous, non-absorbing medium with phase velocity $c$ (wave speed)
- In vacuum, subscript 0 (zero) is used, e.g., $c_0, \mu_0, \varepsilon_0$
- The index of refraction defined as:
  $$n = \frac{c_0}{c}$$
I-1 Wave Optics: definitions

- Wavefunction is a **real** function of position - defined by position vector $\vec{r}$ and time $t$: $u(\vec{r}, t)$

- Wavefunction satisfies the wave equation:

$$\nabla^2 u(r, t) - \frac{1}{c^2} \frac{\partial^2 u(r, t)}{\partial t^2} = 0$$

- Principle of superposition applies:

$$u(\vec{r}, t) = u_1(\vec{r}, t) + u_2(\vec{r}, t)$$
I-1 Wave Optics: definitions

- **OPTICAL INTENSITY** is the optical power per unit surface area (W.cm⁻²). It is the measurable quantity.

- It is proportional to the time average of $u^2(\vec{r},t)$:

  \[ I(\vec{r},t) = 2\langle u^2(\vec{r},t) \rangle_{\Delta t} \]

- $\Delta t$ is taken over many light cycles……
I-1 Wave Optics: definitions

- **OPTICAL POWER** \( P = \text{power (W)} \) flowing into an area \( A \) normal to the direction of propagation:

\[
P(t) = \int_A I(\vec{r}, t) dA
\]

- **OPTICAL ENERGY**: time integral of optical power over the time interval

\[
P = \int_{\Delta t} P(t) dt
\]
I-1 Wave Optics: definitions

- **FLUENCE** = Optical energy per unit surface area (J.cm\(^{-2}\)). Commonly specified for laser light at the focus of a converging lens.

- **Photodetectors:**
  - Photoelectric detectors: photon releases an electron (photocurrent). Photodiode (p-i-n), Schottky diodes (metal-semiconductors), Photomultiplier tubes. **Sensitive to intensity of incident light**
  - Conversion of photon energy into heat: Power meters. Temperature rise is measured with a thermopile. Sensitive to total power absorbed
I-2 Wave Optics: monochromatic waves

- Monochromatic waves have a harmonic (sine, cosine) time dependence:
  \[ u(\vec{r}, t) = a(\vec{r}) \cos[2\pi\nu t + \varphi(\vec{r})] \]

- \( a(\vec{r}) \): Amplitude [V.m\(^{-1}\)]
- \( \varphi(\vec{r}) \): Phase (in radians) (determined by initial conditions)
- \( \nu \) (nu): Frequency (Hz)
- \( \omega \) (omega): Angular frequency (in rads\(^{-1}\)) = \( 2\pi\nu \)
I-2 Wave Optics: monochromatic waves

- It is convenient to use a complex wavefunction function instead:

\[ U(\vec{r}, t) = a(\vec{r})e^{i[2\pi vt + \varphi(\vec{r})]} \]

- From above definition:

\[ u(\vec{r}, t) = \text{Re}[U(\vec{r}, t)] \quad (\text{Re} = \text{real part}) \]

\[ u(\vec{r}, t) = \frac{1}{2}[U(\vec{r}, t) + U^*(\vec{r}, t)] \]
I-2 Wave Optics: monochromatic waves

- Can be rewritten in the form:
  \[ U(\vec{r}, t) = U(\vec{r})e^{2\pi i\nu t} \]

- The amplitude is now a complex function:
  \[ U(\vec{r}) = a(\vec{r})e^{i\phi(\vec{r})} \]

- Helmholtz equation obtained (after substitution into wave equation):
  \( (\nabla^2 + k^2)U(\vec{r}) = 0 \)
  
  \[ k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \text{wavenumber (m}^{-1}) \]
I-2 Wave Optics: monochromatic waves

- Notes:
  - The choice $\cos[2\pi vt + \varphi(\vec{r})]$ is arbitrary; depends on the initial conditions
  - $\sin[\varphi(\vec{r}) - 2\pi vt]$ Would be also an acceptable function

- Most optical phenomena are steady-state (no time dependence): it is therefore often customary to drop the time factor or dependency: $e^{2\pi i vt}$
I-2 Wave Optics: monochromatic waves

- The optical intensity: \( I(\vec{r}) = |U(\vec{r})|^2 \)
- The intensity does not vary with time
- Surfaces of equal phase are called wavefronts:

\[ \varphi(\vec{r}) = \text{constant} \]

Typically: \( \varphi(\vec{r}) = 2\pi q \) (\( q \) is an integer)
I-3 Wave Optics: Elementary waves

- There are various possible solutions of the Helmholtz equation in a homogeneous medium:
  - PLANE WAVE
  - SPHERICAL WAVE
  - PARAXIAL WAVES (GAUSSIAN BEAM - OPTICAL RESONATOR)
I-3 Wave Optics: Plane wave

- The Plane Wave with complex amplitude:
  \[ U(\vec{r}) = Ae^{-i\vec{k} \cdot \vec{r}}, \quad \varphi(\vec{r}) = \vec{k} \cdot \vec{r} \]

- A is the complex envelope and \( k \) is the wave vector, with \( \vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z = \text{constant} \)

- Equation describing parallel planes separated by a distance of one wavelength:
  \[ \lambda = \frac{2\pi}{k} \]
I-3 Wave Optics: Plane wave

- Can choose z axis in the direction of k:

\[ U = A e^{-ikz} \]

\[ u(\vec{r}, t) = |A| \cos[2\pi vt - kz + \arg\{A\}] \]

\[ u(\vec{r}, t) = |A| \cos\left[2\pi \nu \left(t - \frac{z}{c}\right) + \arg\{A\}\right] \]

- c and \( \lambda \) are the phase velocity and wavelength in the medium:

\[ c = \frac{c_0}{n} \quad \text{and} \quad \lambda = \frac{\lambda_0}{n} \]
I-3 Wave Optics: Spherical wave

- The complex amplitude is: 
  \[ U(r) = \frac{A}{r} e^{-ikr} \]

- \( r \) is the radial distance from origin

- Optical Intensity: 
  \[ I(r) = \frac{|A|^2}{r^2} \]

- If \( A \) is real, ie \( \text{arg}\{A\} = 0 \), the surfaces of equal phase: \( kr = 2\pi n \) or \( r_x^2 + r_y^2 + r_z^2 = \left( \frac{2\pi n}{k} \right)^2 \) define concentric spheres, separated by a distance of \( \frac{2\pi}{k} \)

- Large \( r \) becomes plane
I-3 Wave Optics: Spherical wave

- At points close to the z axis and far from the origin:
  - Paraboloidal wave: approximation for behaviour between spherical and planar.
  - At large z, behaviour is almost planar.

- This is typically the behaviour of paraxial waves (eg. the Gaussian beam often found in laser systems)
I-3 Wave Optics: Paraxial waves

- Wavefronts normal are paraxial rays:

Wavefunction of paraxial wave at points along the $z$ axis

Wavefronts and wavefront normals

$\theta_{\text{small}} \approx \sin \theta = \tan \theta$
I-3 Wave Optics: Paraxial waves

- To construct a paraxial wave: start with a plane wave $Ae^{-ikz}$ and modulate the complex envelope $A$ making it a slowly varying function of $r$:

$$U(\vec{r}) = A(\vec{r})e^{-ikz}$$

$A(\vec{r})$ variation with position is very small over a distance of one $\lambda$. It is still approximately planar.
I-3 Wave Optics: Paraxial waves

- Paraxial waves satisfy the paraxial Helmholtz equation:

\[ \nabla^2_T A - i2k \frac{\partial A}{\partial z} = 0 \]

\[ \nabla^2_T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \text{Transverse Laplace operator} \]

- Most useful is the Gaussian beam (mode of the spherical-mirror resonator)
II - Electromagnetic Optics

- Light is an electromagnetic phenomenon: carries electric $\vec{E}(\vec{r},t)$ and magnetic fields $\vec{H}(\vec{r},t)$

- These are vector waves: scalar wave equation fails to explain electric and magnetic effects induced by light

- Problem: how can we describe the electromagnetic state of matter in the presence of light?
II - Electromagnetic Optics: Definitions

- New set of vectors is required to describe the response of matter:
  - Electric current density $\vec{j}$
  - Electric displacement (electric flux density) $\vec{D}$
  - Magnetic displacement (magnetic induction) $\vec{B}$
  - $\rho$ density of free charges

- $E, H, B, D, j$ and $\rho$ are related by Maxwell’s equations (set of 4 coupled PDE’s)
II - Electromagnetic Optics: Definitions

- General solution of Maxwell’s equations is complicated (would provide electromagnetic response of matter - \(D\) and \(B\) - in the presence of \(E\) and \(H\) fields)

- For harmonic fields and isotropic media, relation between applied fields and response is simple
II - Electromagnetic Optics: In Vacuo

- $\varepsilon = \text{Electric permittivity or dielectric constant}$
  $$\vec{D} = \varepsilon \vec{E}$$

- $\mu = \text{magnetic permeability}$
  - $\mu \sim 1$ non-magnetic (most substances)
  - $\mu > 1$ paramagnetic
  - $\mu < 1$ diamagnetic
  $$\vec{B} = \mu \vec{H}$$

- $\sigma = \text{specific conductivity}$
  - $\sigma$ negligibly small: insulators (dielectrics)
  - $\sigma$ not negligibly small: conductors
  $$\vec{j} = \sigma \vec{E}$$
II - Electromagnetic Optics: Definitions

- Previous set of equations describes the response of matter in the presence of weak fields.
- Linear response: $1^{st}$ power of fields
- For strong fields (strength of the order of valence electrons binding energies):
  - Response is non linear
  - Must include higher-order components of the fields
The laws of Optics must be modified (Non-linear Optics, Bloembergen, 1965)
Effects of the fields can be described using “additive” relations:

\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \]
\[ \vec{P} = \text{Polarization} = \text{Dipole moment/m}^3 \]
\[ \vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} \]
\[ \vec{M} = \text{Magnetization} = \text{Magnetic moment/m}^3 \]
II - Electromagnetic Optics: Definitions

- For weak fields, polarization and magnetization are assumed to be linearly proportional to the applied fields:

\[\vec{P} = \chi \varepsilon_0 \vec{E}\]
\[\chi = \text{electric susceptibility}\]
\[\vec{D} = \varepsilon_0 \vec{E} + \chi \varepsilon_0 \vec{E} = \varepsilon_0 (1 + \chi) \vec{E}\]
\[\varepsilon = \varepsilon_0 (1 + \chi)\]
\[\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1 + \chi)\]
\[\text{relative permittivity}\]

\[\mu_0 \vec{M} = \mu_0 \chi_m \vec{H}\]
\[\chi_m = \text{magnetic susceptibility}\]
\[\vec{B} = \mu_0 \vec{H} + \chi_m \mu_0 \vec{H} = \mu_0 (1 + \chi_m) \vec{H}\]
\[\mu = \mu_0 (1 + \chi_m)\]
\[\mu_r = \frac{\mu}{\mu_0} = (1 + \chi_m)\]
\[\text{relative permeability}\]
II - Electromagnetic Optics: Maxwell’s Equations

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \]
\[ \nabla \cdot \vec{D} = \rho \]

\[ \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{j} \]
\[ \nabla \cdot \vec{B} = 0 \]

- In optics, generally non-magnetic media and no currents (\( \vec{M} = 0 \) and \( \vec{j} = 0 \))

- The flow of electromagnetic energy is given by the Poynting vector: \( \vec{P} = \vec{E} \times \vec{H} \)
II - Electromagnetic Optics: Maxwell’s Equations

- Most optical materials are dielectrics:
  - \( L = \) linear: if \( P \) is linearly related to \( E \)
  - \( ND = \) non-dispersive: instantaneous response: \( P \) at \( t \) is determined by \( E \) at \( t \).
  - \( H = \) homogeneous: relation between \( P \) and \( E \) is independent of \( r \)
  - \( I = \) isotropic: relation between \( P \) and \( E \) is independent of the direction of \( E \). Medium is identical from all directions of space.
II - Electromagnetic Optics: Maxwell’s Equations

- Medium is L, ND, H and I:
  \[ \vec{P} = \chi \varepsilon_0 \vec{E}; \vec{D} = \varepsilon \vec{E}; \varepsilon = \varepsilon_0 (1 + \chi) \]

- Each component of E, H satisfy separately the wave equation (same as wave optics):
  \[ \nabla^2 u(r,t) - \frac{1}{c^2} \frac{\partial^2 u(r,t)}{\partial t^2} = 0 \text{ with } c = \frac{1}{(\varepsilon \mu_0)^2} = \frac{c_0}{n} \]
  \[ n = \left( \frac{\varepsilon}{\varepsilon_0} \right)^{\frac{1}{2}} = (1 + \chi)^{\frac{1}{2}} \]
II - Electromagnetic Optics: Maxwell’s Equations - inhomogeneous medium

- Medium is L, ND, I, inhomogeneous
- (e.g. a graded-index optical fibre)
- The spatial variations of $n = n(\mathbf{r})$ are small over distances of a few wavelengths

\[ \mathbf{P} = \chi(\mathbf{r}) \varepsilon_0 \mathbf{E}; \quad \mathbf{D} = \varepsilon(\mathbf{r}) \mathbf{E} \]

\[ \nabla^2 \mathbf{E} - \frac{1}{c(\mathbf{r})^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \]
II - Electromagnetic Optics: Maxwell’s Equations

- Medium is L, ND, H but anisotropic: relation between $P$ and $E$ depends on the direction of $E$

- $P$ and $E$ are not necessarily parallel:
  - Dielectric properties described by an array of $(3\times3)$ constants called the susceptibility tensor
II - Electromagnetic Optics: Maxwell’s Equations-Anisotropic medium

- Each component of $P$ (or $D$) is given by:

$$P_i = \sum_j \varepsilon_0 \chi_{ij} E_j$$

$i, j = 1, 2, 3$ denotes $x, y, z$ components

$$D_i = \sum_j \varepsilon_{ij} E_j$$

$\varepsilon_{ij}$ components of electric permittivity tensor

- Typically crystals with non cubic symmetries are anisotropic media
II - Electromagnetic Optics: Maxwell’s Equations nonlinear medium

- The relation between $P$ and $E$ is non linear: $\vec{P} = \Psi(\vec{E})$, e.g. $\vec{P} = a_1 \vec{E} + a_2 \vec{E}^2 + a_3 \vec{E}^3$

- Maxwell’s equations must be used to derive a non-linear partial differential eqn

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2} = \mu_0 \frac{\partial^2 \Psi(\vec{E})}{\partial t^2}$$

Basic equation of non linear optics
II - Electromagnetic Optics: Elementary EM waves

- The Transverse Electromagnetic (TEM) Plane Wave (medium L,H,I):

\[
\begin{align*}
\vec{E}(\vec{r}) &= \vec{E}_0 e^{-i\hat{k}\cdot\vec{r}} \\
\vec{H}(\vec{r}) &= \vec{H}_0 e^{-i\hat{k}\cdot\vec{r}}
\end{align*}
\]

(1) From Maxwell:
\[
\left(\frac{E_0}{H_0}\right) = \left(\frac{\omega \mu_0}{k}\right) = \left(\frac{c_0 \mu_0}{n}\right) = \left(\frac{\varepsilon_0}{\eta}\right) = \frac{1}{\eta}
\]

\[\eta = \text{(optical) impedance of medium}\]

(2) From Poynting:
\[I = \frac{|E_o|^2}{\eta}\]
III-Polarisation of Light

- Polarisation = time course of the direction of the electric field vector $E(r,t)$
- In paraxial optics, EM waves are approximately TEM: $E(r,t)$ lies in transverse plane
- If medium is isotropic: wave is elliptically polarized
III-Polarisation of Light

- Polarisation plays an important role in optics:
  - Amount of reflected light depends on polarisation state at the boundary (interface)
  - Amount of light absorbed depends on state of polarisation (dichroism)
  - Refractive index of anisotropic materials depends on polarisation state (see optical devices - birefringent materials)
  - Rotation of plane of polarisation of linearly polarised light in presence of external electric or magnetic field
III-Polarisation of Light: polarisation ellipse

- \( \vec{E}(z,t) = \text{Re}\left[ \vec{A} e^{-i 2\pi \nu (t - \frac{z}{c})} \right] \)

Monochromatic plane wave travelling in Oz direction with velocity \( c \)

- Complex envelope (amplitude): \( \vec{A} = A_x \hat{x} + A_y \hat{y} \)
  
  \[
  A_x = a_x e^{-i\phi_x}; \quad A_y = a_y e^{-i\phi_y}
  \]
III-Polarisation of Light: polarisation ellipse

- Polarisation = End point of $E(z,t) =$ location of points whose coordinates are $(E_x, E_y)$: 

$$\vec{E}(z,t) = E_x \hat{x} + E_y \hat{y}$$

Defining $\tau = 2\pi \nu \left( t - \frac{z}{c} \right)$

$$E_x = a_x \cos(\tau + \varphi_x), \quad E_y = a_y \cos(\tau + \varphi_y), \quad E_z = 0$$

$$\frac{E_x}{a_x} = \cos \tau \cos \varphi_x - \sin \tau \sin \varphi_x$$ etc...and $\varphi = \varphi_y - \varphi_x$
III-Polarisation of Light: polarisation ellipse

- Equation of ellipse (conic):

\[
\left( \frac{E_x}{a_x} \right)^2 + \left( \frac{E_y}{a_y} \right)^2 - 2\left( \frac{E_x}{a_x} \frac{E_y}{a_y} \right) \cos \varphi = \sin^2 \varphi
\]

The "tilt" $\Psi$ is obtained from:

\[
\tan 2\Psi = \frac{2a_x a_y \cos \varphi}{(a_x^2 - a_y^2)}
\]
III-Polarisation of Light: polarisation ellipse

- The magnetic vector is also elliptically polarised
- At fixed value of z, $\mathbf{E}$ rotates at frequency ($\nu$) $\nu$ in (x-y) plane tracing out an ellipse
- At fixed t (snap shot): the location of the tip follows a helical trajectory
- State of polarisation determined by tilt (value of psi) and ratio of major to minor axes
III-Polarisation of Light: Polarisation ellipse

Timecourse of tip of $E$ is an elliptical helix:
III-Polarisation of Light: polarisation ellipse

Right-Handed Elliptically Polarised: \( \sin \phi > 0 \)

Left-Handed Elliptically Polarised: \( \sin \phi < 0 \)
III-Polarisation of Light: polarisation ellipse

- The nature of the polarisation can be determined from:

\[
\frac{E_y}{E_x} = \frac{a_y}{a_x} e^{i(\phi_x - \phi_y)} = \frac{a_y}{a_x} e^{-i\phi}
\]

- Linear Polarisation:

\[
\frac{E_y}{E_x} = (-1)^m \frac{a_y}{a_x}, \text{ as ellipse reduces to a straight line when } \phi = m\pi (m = 0, \pm 1, \pm 2, \ldots)
\]

Linear polarisation also for \(a_x\) or \(a_y = 0\)
III-Polarisation of Light: polarisation ellipse

- Circular Polarisation: the ellipse degenerates into a circle if $a_x = a_y = a_0$ and $\varphi = m\pi/2$ ($m = \pm 1, \pm 3, \pm 5, \ldots$)

$$E_x^2 + E_y^2 = a_0^2$$

- Using complex form:

  **Right-handed circularly polarized**: $a_x = a_y, \varphi = \pi/2$

  $$\frac{E_y}{E_x} = e^{-i\frac{\pi}{2}} = -i$$

  **Left-handed circularly polarized**: $a_x = a_y, \varphi = -\pi/2$

  $$\frac{E_y}{E_x} = e^{i\frac{\pi}{2}} = i$$
III-Polarisation of Light: Matrix Representation; Jones Vector

- A monochromatic plane wave is completely determined by the knowledge of the complex envelope $A_x$ and $A_y$
- Can be represented in the form of a 2-component column matrix - the Jones vector:

$$\vec{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$
III-Polarisation of Light: Jones Vector

- From J, one can calculate the total light intensity:

\[
I = \left( |A_x|^2 + |A_y|^2 \right) / 2 \eta
\]

- The orientation and shape of the polarisation ellipse can be obtained from:

\[
\frac{a_y}{a_x} = \frac{|A_y|}{|A_x|}; \varphi = \varphi_y - \varphi_x = \text{arg}\{A_y\} - \text{arg}\{A_x\}
\]
III-Polarisation of Light: Jones Vector

- Jones vectors for typical polarisations:
  intensity is normalised so that:

\[
\left( |A_x|^2 + |A_y|^2 \right) = 1 \text{ and } \varphi_x = 0
\]

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Linear Polarisation in x direction

\[
\begin{bmatrix}
\cos \psi \\
\sin \psi
\end{bmatrix}
\]

Linear Polarisation making angle \( \psi \) with x axis

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
1 \\
(-i)
\end{bmatrix}
\]

Right(left)-handed circular polarisation
III-Polarisation of Light: Jones Matrix

- Jones vectors $J_1$ and $J_2$ are orthogonal if (inner product is 0):
  \[
  J_1 \cdot J_2^* = (A_{1x}A_{2x}^* + A_{1y}A_{2y}^*) = 0
  \]

- Any arbitrary Jones vector $J$, can be analysed as a weighted superposition of two orthogonal polarisations:

\[
J = \alpha_1 J_1 + \alpha_2 J_2
\]

\[
\alpha_1 = J \cdot J_1^*; \quad \alpha_2 = J \cdot J_2^*
\]

$J_1, J_2$ normalised to unity

\[
J_1 \cdot J_1^* = J_2 \cdot J_2^* = 1
\]
III-Polarisation of Light: Jones Matrix

- A linear optical system that maintains the plane wave nature of light but alters its polarisation can be represented by a $(2 \times 2)$ Jones matrix $T$:

\[
\begin{align*}
\vec{J}_1 &= \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix} & T &= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} & \vec{J}_2 &= \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} \\
\text{Input Wave} & & \text{Optical System} & & \text{Output Wave} \\
\end{align*}
\]

\[
\vec{J}_2 = T\vec{J}_1
\]
### III-Polarisation of Light: Jones Matrix

<table>
<thead>
<tr>
<th>Examples of Jones matrices:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. The Linear Polariser:</strong></td>
</tr>
<tr>
<td>[ T = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{pmatrix} ]</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>2. The Wave Retarder:</strong></td>
</tr>
<tr>
<td>[ T = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; e^{-i\Delta} \end{pmatrix} ]</td>
</tr>
<tr>
<td>[ \Delta = \frac{\pi}{2} = \text{quarter-wave retarder} ]</td>
</tr>
<tr>
<td>[ \Delta = \pi = \text{half-wave retarder} ]</td>
</tr>
<tr>
<td>[ \begin{pmatrix} A_{1x} \ A_{1y} \end{pmatrix} \rightarrow \begin{pmatrix} A_{2x} = A_{1x} \ A_{2y} = A_{1y} e^{-i\Delta} \end{pmatrix} ]</td>
</tr>
<tr>
<td><strong>3. The Polarisation Rotator:</strong></td>
</tr>
<tr>
<td>[ T = \begin{pmatrix} \cos \theta &amp; -\sin \theta \ \sin \theta &amp; \cos \theta \end{pmatrix} ]</td>
</tr>
<tr>
<td>[ \begin{pmatrix} \cos \theta_1 \ \sin \theta_1 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta_2 \ \sin \theta_2 \end{pmatrix} ]</td>
</tr>
<tr>
<td>[ \theta_2 = \theta + \theta_1 ]</td>
</tr>
</tbody>
</table>
III-Polarisation of Light: Normal modes

- Normal modes of a polarisation system are the states of polarisation that remain unchanged when transmitted through the system.
- Normal modes = eigenvectors of $\mathbf{T}$ matrix (2 modes)

$$\mathbf{T}\mathbf{\bar{J}} = \mu\mathbf{\bar{J}}$$
III-Polarisation of Light: Normal modes

- Normal modes are orthogonal and form a basis set (T is hermitian)
- Any input wave $\mathbf{J} = \text{superposition of normal modes}$: $\mathbf{J} = \alpha_1 \mathbf{J}_1 + \alpha_2 \mathbf{J}_2$
- The response can be easily evaluated using:

$$T\mathbf{J} = T(\alpha_1 \mathbf{J}_1 + \alpha_2 \mathbf{J}_2) = \alpha_1 T\mathbf{J}_1 + \alpha_2 T\mathbf{J}_2 = \alpha_1 \mu_1 \mathbf{J}_1 + \alpha_2 \mu_2 \mathbf{J}_2$$

- Problem: Find the Normal Modes
III-Polarisation of Light: Example of normal modes

- Reflection and refraction of monochromatic plane wave of arbitrary polarisation incident at dielectric boundary \((n_1,n_2)\)

The normal modes (from Maxwell’s) are the two linear polarisations:

\(T_E\) (transverse electric, parallel to the boundary): sigma or s polarisation

\(T_M\) (Transverse magnetic parallel to the plane of incidence): parallel or pi polarisation
IV-Crystal Optics

- Crystals are anisotropic media: electric displacement vector $D$ depends (possibly) on all the components of applied $E$ field.

- Each component of $D$ can be written as:

$$D_i = \sum_j \varepsilon_{ij} E_j \text{ with } i, j = 1, 2, 3 \equiv x, y, z$$

$\varepsilon$ is a second-rank tensor: the permittivity tensor

Electric displacement $\vec{D}$ is the contraction of a 2-tensor and a vector (tensor rank one): $\vec{D} = \vec{\varepsilon} \vec{E}$
IV-Crystal Optics

- Examples of anisotropic media

(a) Completely isotropic: long and short-range disorder

(b) Short-range order, long-range disorder: average macroscopic behaviour is isotropic

(c) Positional and orientational orders: anisotropic (except fcc lattices)

(d) Short-range disorder, long-range order: average macroscopic behaviour is anisotropic
IV-Crystal Optics

- There always exists a system of coordinates in which $\varepsilon$ has only diagonal elements: $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$
- This system defines the Principal Axes: directions of space for which $E$ and $D$ are parallel.
- The principal refractive indices are:

\[
\begin{align*}
n_1 &= \left( \frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} \\
n_2 &= \left( \frac{\varepsilon_2}{\varepsilon_0} \right)^{1/2} \\
n_3 &= \left( \frac{\varepsilon_3}{\varepsilon_0} \right)^{1/2}
\end{align*}
\]
Anisotropy leads to birefringence: phase velocity of an optical beam clearly depends on the direction of polarisation of its E vector.

Three types of crystals:

- Uniaxial: \( n_1 = n_2 = n_o \) (ordinary index), \( n_3 = n_e \) (extraordinary index) calcite, quartz
- Biaxial: \( n_1, n_2, n_3 \) are all different.
- Isotropic \( n_1=n_2=n_3 \)
Geometrical construction that completely describes the optical properties: it specifies the values of the Principal refractive indices and the directions of the Principal axes.

This is called the Index Ellipsoid. It is the surface of equation:

$$\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1$$

$x, y, z$: principal axes

$n_1, n_2, n_3$: principal indices
IV-Crystal Optics

1. Index ellipsoid is an ellipsoid of revolution for uniaxial crystals

2. Index ellipsoid is a sphere for cubic crystal

3. $z$ is called optic axis for uniaxial crystals
IV-Crystal Optics

- Propagation of plane EM waves (linearly polarised) along one of Principal axes: what are the normal modes?

- Linear polarisation along x or y directions: Wave travels at phase velocity \( c_0/n_1 \) or \( (c_0/n_2) \) without change of polarisation.

\[
D_1 = \varepsilon_1 E_1 \quad (D_2 = \varepsilon_2 E_2)
\]

If \( k \) is along Oz, the Normal modes are the linearly polarised waves in the x and y directions respectively.
If \( k \) is along Oz, but \( E \) is in x-y plane making angle \( \theta \) with Ox

- Resulting polarisation?
IV-Crystal Optics

- If $k$ is along Oz, but $E$ is in x-y plane making angle $\theta$ with Ox
- Traveling wave is a sum of the normal modes: each travels at $(c_0/n_1)$ and $(c_0/n_2)$ respectively
- The phase difference after a distance $d$ travelled through the crystal:

\[ \varphi = \frac{2\pi}{\lambda_0} (n_2 - n_1) d \]

- The output wave is elliptically polarised.
- Crystal acts as a wave retarder
- Retardation plates are polarisation state converters
IV-Crystal Optics

- Propagation in any arbitrary direction (take the case of uniaxial crystals only): \( k \) makes angle \( \theta \) with respect to \( O_z \) (optic axis).

- The normal modes are linearly polarised and orthogonal directions \( OA \) and \( OB \) (next slide).

- They form the semi-axes of the **Index Ellipse** and define the **Ordinary** (Direction \( OB \)) and **Extraordinary** (Dir. \( OA \)) waves respectively.

- \( O \) (Ordinary) wave travels at \( c_0/n_O \), E wave travels at \( c_0/n_e(\theta) \) (uniaxial crystal).

- Simple geometry is used to calculate \( n_e(\theta) \).
**IV-Crystal Optics**

\[ \sin(\theta) = \frac{z}{n_e(\theta)} \]

\[ \sin(90 - \theta) = \cos(\theta) = \frac{y}{n_0(\theta)} \]

**Equation of ellipse**

\[ \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} = 1 \]
For $\theta = 0$ (k along z), $n_0 = n_e(\theta)$ so that there is no birefringent behaviour (Hence the name uniaxial).

A Retardation plate has its optic axis in the plane of the plate surface. The desired state of polarisation is obtained by adjusting the thickness (see p.64)
V POLARIZATION DEVICES: POLARIZERS

- Linear polarizer:
  - Transmits components of E field along the direction of its transmission axis
  - Blocks the orthogonal component
  - Can be achieved by:
    - Dichroic materials (selective absorption); Polaroid sheet
    - Selective reflection from isotropic media; Brewster’s angle
    - Selective reflection/refraction in anisotropic...
V POLARIZERS: POLARIZING BEAMSPLITTERS

- Ordinary and extraordinary waves refract at different angles in anisotropic crystal: polarized light can be obtained from unpolarized light.
- Typically two cemented prisms made of uniaxial materials with different orientations:
  - Wollaston prism
  - Rochon prism
V POLARIZERS: POLARIZING BEAMSPRITTERS

O-ray is totally internally reflected at cement interface.
V POLARIZERS: Wave Retarders

- Convert one polarisation into another
- Normal modes are linearly polarised along the fast $n_f$ and slow $n_s$ axes.
- Constructed from anisotropic materials in the form of plates: light is made to travel along one of the principal axis
- Retardation is directly proportional to plate thickness:
  \[ \Gamma = \frac{2\pi}{\lambda} (n_f - n_s) d \]
V POLARIZERS: Wave Retarders

- Retardation is directly proportional to the thickness of the plate.
- Retardation is inversely proportional to the wavelength.
- Thin sheet of mica:
  - Indices: 1.599 and 1.594 at 633 nm (He-Ne laser) → $\Gamma / d \approx 15.8$ rad/mm
  - Sheet of 63.3 microns yields $\Gamma \approx \pi$ rad.
V Wave Retarders: Light intensity control

- Wave retarder placed between 2 cross-polarisers whose axes are at 45 deg. with respect to the axes of the retarder.

- Intensity transmittance of this device is:

\[ I_T = \sin^2\left(\frac{\Gamma}{2}\right) \]

- Intensity can be changed by altering the retardation (see Electro-Optics) via use of electro-optic anisotropic crystals.