PS403 - Digital Signal processing

IV. DSP - The Z-Transform

Key Text:

Digital Signal Processing with Computer Applications (2nd Ed.)

Paul A Lynn and Wolfgang Fuerst, (Publisher: John Wiley & Sons, UK)

We will cover in this section

How to compute the Z-Transform of a signal/ impulse response

Poles and zeros of the Z-Transform of a signal or LTI processor

Frequency analysis using the Z-Transform

The Z-Transform

Definition: $$X(Z) = \sum_{n=0}^{\infty} x[n] Z^{-n}$$

Origins:

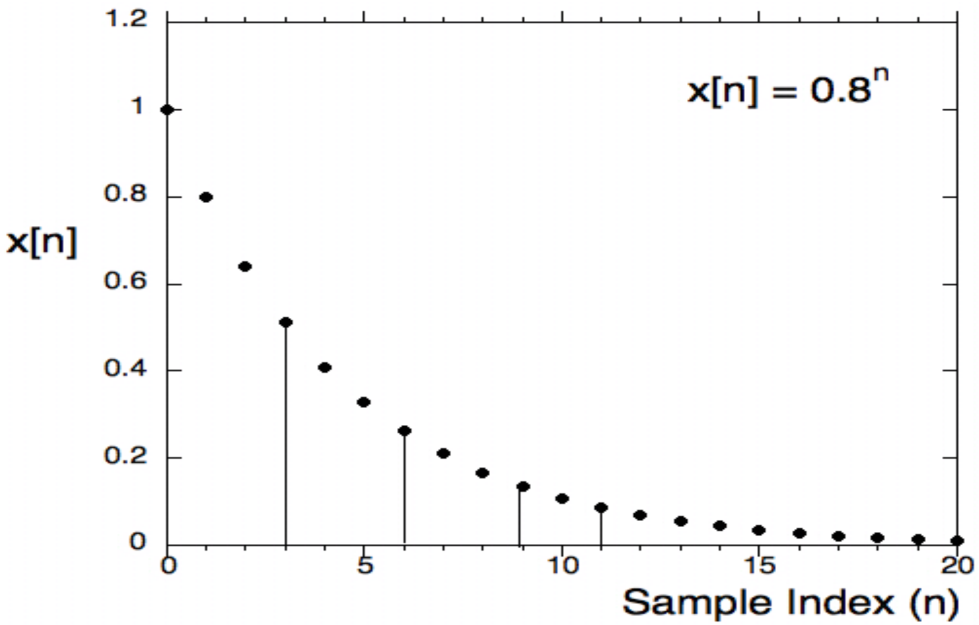
The Z-Transform is useful for looking at signals and systems in the frequency domain. It is not unlike the Laplace/Fourier Transforms except that it has its origins in sampled data systems, while Laplace/ Fourier Transforms apply to analog (continuous valued) signals and systems.

X(Z) is NOT concerned with x[n] prior to n = 0; Unilateral $\sum_{n=0}^{n=\infty}$

X(Z) is effectively a power series in Z^{-1}, with coefficients given by the values of x[n].

The Z-Transform

Example: Find the Z-T of $x[n] = 0.8^0, 0.8^1, 0.8^2. 0.8^3,....$



$$X(z) = 1 \times Z^0 + 0.8 \times Z^{-1} + 0.64 \times Z^{-2} + 0.512 \times Z^{-3} + ...$$

$$= 1 + (0.8 \times Z^{-1})^1 + (0.8 \times Z^{-1})^2 + (0.8 \times Z^{-1})^3 + ...$$

$$\boxed{\Rightarrow X(Z) = \frac{1}{1 - 0.8Z^{-1}} = \frac{Z}{Z - 0.8}}$$

The Z-Transform

Example: Find the signal x[n] with Z-Transform: $X(Z) = \dfrac{1}{Z+1.2}$

$$X(Z) = \frac{1}{Z+1.2} = \frac{Z^{-1}}{1+1.2.Z^{-1}}$$

$$= Z^{-1}\left(1+1.2Z^{-1}\right)^{-1}$$

$$= Z^{-1}\left[1+\left(-1.2Z^{-1}\right)^{1}+\left(-1.2Z^{-1}\right)^{2}+\left(-1.2Z^{-1}\right)^{3}+..\right]$$

$$= 1.Z^{-1} - 1.2Z^{-2} + 1.44Z^{-3} + \ldots\ldots\ldots$$

Coeff of x[1]

Coeff of x[2]

Coeff of x[3]

By inspection:
x[0] = 0
x[1] = 1
x[2] = -1.2
x[3] = 1.44
$x[n] = (-1.2)^{n-1}$

The Z-Transform

1. Notice that although both signals theoretically contain an infinite number of sample values, their Z-Ts are very compact

2. One can think of the of Z as a time-shift operator
Multiplication by Z ≡ time advance by one sampling interval
Division by Z ≡ time delay by one sampling interval

Example: Time shifting a unit impulse

The Z-T of a unit impulse is given by:

$$X(Z) = \sum_{n=0}^{\infty} \delta[n] Z^{-n} = Z^{-n}\Big|_{n=0} = 1$$

The Z-Transform

The Z-T of a unit impulse delayed by n_0 sampling intervals is:

$$X(Z) = \sum_{n=0}^{\infty} \delta[n - n_0] Z^{-n} = Z^{-n}\Big|_{n=n_0} = Z^{-n_0}$$

Hence time shifting in Z-space becomes a simple operation - multiplication by Z^{-n_0} for time delay or Z^{+n_0} for time advance

As the Z-T & F-T are related, the convolution theorem applies !

$$X(Z) = \sum_{n=0}^{\infty} x[n] Z^{-n}$$, let Z = exp(jΩ)

$$\Rightarrow X(\Omega) = \sum_{n=0}^{n=\infty} x[n] \exp(-jn\Omega)$$

The Z-Transform

If x[n] is an input signal and h[n] a processor response; we know that:

$$y[n] = x[n]*h[n]$$

and hence

$$Y(\Omega) = X(\Omega).H(\Omega)$$

and

$$Y(Z) = X(Z).H(Z)$$

Convolution Theorem applies to Z-Ts

The Z-Transform

Convolution Theorem - *proof by example*

Direct convolution: y[n] = h[n]*x[n]

n	0	1	2	3	4	5	6	7	8
x[n]	1	-2	3	-1	-1	0	0	0	0
h[n]	2	1	-1	0	0	0	0	0	0
y[n]	2	-3	3	3	-6	0	1	0	0

Z-T Multiplication: Y(Z) = H(Z).X(Z)

$$X(Z) = \sum_{n=0}^{\infty} x[n] Z^{-n}$$

$X(Z) = 1 - 2Z^{-1} + 3Z^{-2} - Z^{-3} - Z^{-4}......$

$H(Z) = 2 + Z^{-1} - Z^{-2}$

$$Y(Z)=H(Z).X(Z)=2 - 3Z^{-1} + 3Z^{-2} + 3Z^{-3} - 6Z^{-4} +0Z^{-5} + Z^{-6}=\sum_{n=0}^{\infty} y[n] Z^{-n}$$

By inspection: y[n] = 2, -3, 3, 3, -6, 0, 1, 0, 0,.......

The Z-Transform

A processor has an impulse response in Z-space given by:

$$H(Z) = \frac{1}{Z(Z-1)(2Z-1)}$$

Find the (i) corresponding difference equation describing the action of the processor and (ii) its impulse response.

(i) find y[n]

$$H(Z) = \frac{1}{Z(Z-1)(2Z-1)} = \frac{Y(Z)}{X(Z)}$$

$$\Rightarrow Y(Z)\left[Z(Z-1)(2Z-1)\right] = X(Z)$$

$$2Z^3Y(Z) - 3Z^2Y(Z) + ZY(Z) = Y(Z)$$

The Z-Transform

Since multiplication by Z is ≡ time advance by one sampling interval we can write:

$$2y[n+3] - 3y[n+2] + y[n+1] = x[n]$$

Since this is a simple recurrence formula we can let n → n-3 without any loss of generalisation

$$\Rightarrow 2y[n] - 3y[n-1] + y[n-2] = x[n-3]$$

or

$$\boxed{y[n] = 1.5y[n-1] - 0.5y[n-2] + 0.5x[n-3]}$$

Show that h[n] = 0, 0, 0, 0.5, 0.75, 0.875,....

The Z-Transform

We can now study more complex processors, e.g.,

$$H(Z) = \frac{Z^2(Z-1)(Z^2+1)}{(Z+0.8)(Z^2+1.38593Z+0.9604)(Z^2-1.64545Z+0.9025)} = \frac{Y(Z)}{X(Z)}$$

$[Z^5 - Z^4 + Z^3 - Z^2].X(Z) = [Z^5 + 0.54048Z^4 - 0.62519Z^3 - 0.66354Z^2 + 0.60317Z + 0.69341].Y(Z)$

x[n+5] - x[n+4] + x[n+3] - x[n+2] =
y[n+5] + 0.54048y[n+4] - 0.62519 y[n+3] - 0.66354y[n+2] + 0.60317y[n+1]+ 0.69341y[n]

Since we can let n $\rightarrow$ n-5 without any loss of generalisation

y[n] = -0.54048y[n-1] + 0.62519 y[n-2] + 0.66354y[n-3] - 0.60317y[n-4] - 0.69341y[n-5] + x[n] - x[n-1] + x[n-2] - x[n-3]

The Z-Transform

Steady State Response of a Processor - Final Value Theorem

Step Response S[n] as 'n' gets very large !

If $x[n] \xrightarrow{Z-Transform} X(Z)$

$$\Rightarrow \lim_{n \to \infty} x[n] = \lim_{Z \to 1} \left(\frac{Z-1}{Z}\right) X(Z)$$

Note that the Z-T of a unit step **u[n]** = $\frac{Z}{Z-1}$

Hence if u[n] is an input signal to a system with transfer function H(Z), then the output signal is given by:

$$Y(Z) = \left(\frac{Z}{Z-1}\right) H(Z)$$

The Z-Transform

Final Value Theorem

$$\Rightarrow \lim_{n \to \infty} S[n] = \lim_{Z \to 1} \left(\frac{Z-1}{Z}\right) Y(Z)$$

$$= \lim_{Z \to 1} \left(\frac{Z-1}{Z}\right)\left(\frac{Z}{Z-1}\right) H(Z) = \lim_{Z \to 1} H(Z)$$

Finally the Final Value Theorem States that:

$$\boxed{\lim_{n \to \infty} S[n] = \lim_{Z \to 1} H(Z)}$$

The Z-Transform

Final Value Theorem: Example

Consider y[n] = 0.8y[n-1] + x[n]

Then h[n] = 0.8h[n-1] + δ[n]

and
$$H(Z) = \frac{Z}{Z-0.8}$$

Ergo
$$\lim_{n \to \infty} \mathrm{S}[n] = \lim_{Z \to 1} H(Z) = \frac{1}{1-0.8} = 5$$

Look back at step response of this processor in section 3

The Z-Transform

Frequency Analysis using Z-Transforms

$$X(Z) = \sum_{n=0}^{n=\infty} x[n] Z^{-n}$$

Substituting Z = exp(jΩ)

$$\Rightarrow X(\Omega) = \sum_{n=0}^{n=\infty} x[n] \exp(-jn\Omega)$$

The Z-Transform is most useful not just for compact description of LTI processors with long/infinite impulse reponse but also for inferring their frequency response

The Z-Transform

Poles and zeros of X(Z):

X(Z) is always a rational function, i.e., it can always be written as a ratio of two polynomials in Z.

Hence on can write:

$$X(Z) = \frac{N(Z)}{D(Z)} = K.\frac{(Z - Z_1).(Z - Z_2).(Z - Z_3)\ldots\ldots\ldots}{(Z - P_1).(Z - P_2).(Z - P_3)\ldots\ldots\ldots}$$

Where	$Z_1, Z_2, \ldots\ldots$	are the Zeros of X(Z)
and	$P_1, P_2, \ldots\ldots$	are the Poles of X(Z)

If $x[n] \in R \Rightarrow$ Poles & Zeros are either real or occur in complex conjugate pairs

The Z-Transform

Z-Plane and the Argand Diagram:

It is often useful to plot the poles and zeros of a Z-T on an Argand diagram. In fact, a trained eye will deduce the salient features of a processor response from the plot quite easily !

Convention

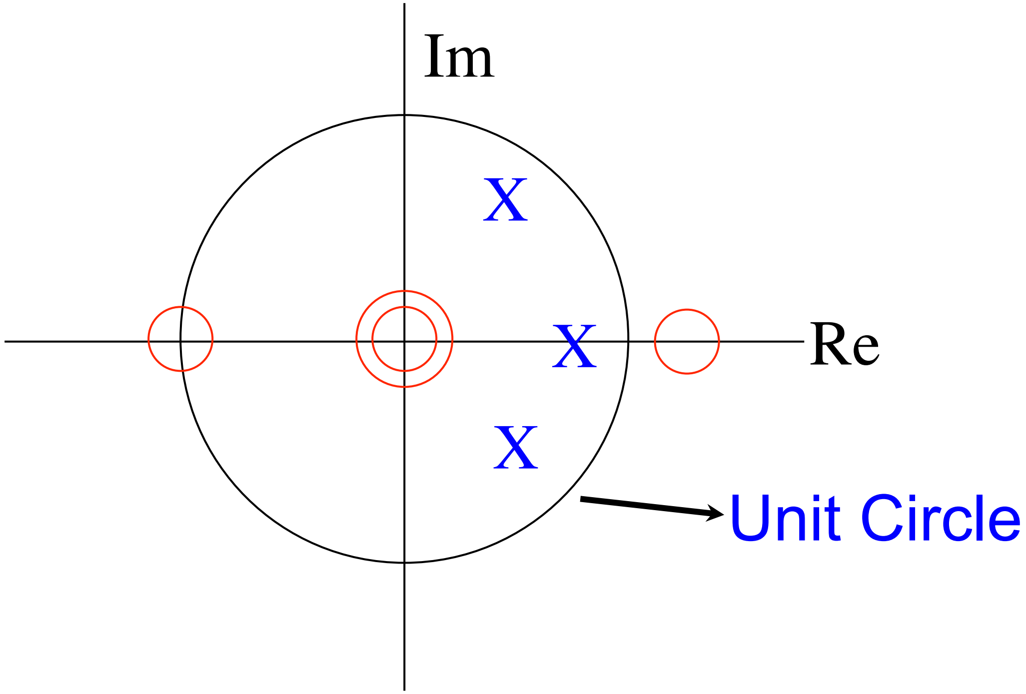
Zeros are plotted as ○ symbols
Poles are plotted as X symbols

Example: Plot the poles and zeros of the following ZT:

$$X(Z) = \frac{Z^2(Z-1.2)(Z+1)}{(Z-0.5+j0.7)(Z-0.5-j0.7)(Z-0.8)}$$

The Z-Transform

Example: Poles and zeros



By inspection we have:

Zeros @ Z= 0 (2 of), Z = 1.2 & Z = -1.0 &

Poles @ Z= 0.8, Z = 0.5-j0.7 & Z = 0.5+j0.7

The Z-Transform

Inferring LTI System Stability from Pole-Zero Plots

Consider a digital LTI processor with ZT: $H(Z) = \dfrac{Y(Z)}{X(Z)}$

Clear that Y(Z) {and concomitantly the processor output y[n]} determines the location of all zeros of the ZT and that X(Z) and {hence x[n]} determines the locations of the poles.

Say that $H(Z) = \dfrac{Y(Z)}{X(Z)} = \dfrac{1}{Z-\alpha}$

System has no zeros and one pole at $Z = \alpha$

$$ZY(Z) - \alpha Y(Z) = X(Z)$$
$$\Rightarrow y[n+1] - \alpha y[n] = x[n]$$
$$\Rightarrow y[n] = \alpha y[n-1] + x[n-1]$$

Find h[n] !

The Z-Transform

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{Z - \alpha}$$

$\Rightarrow$ h[n] = αh[n-1] + δ[n-1]

$\Rightarrow$ h[n] =0, 1,α, α^2, α^3, α^4,............

Hence, if $\alpha < 1$, h[n] may be infinite but at least it is bounded which implies H(Z) describes a stable system.

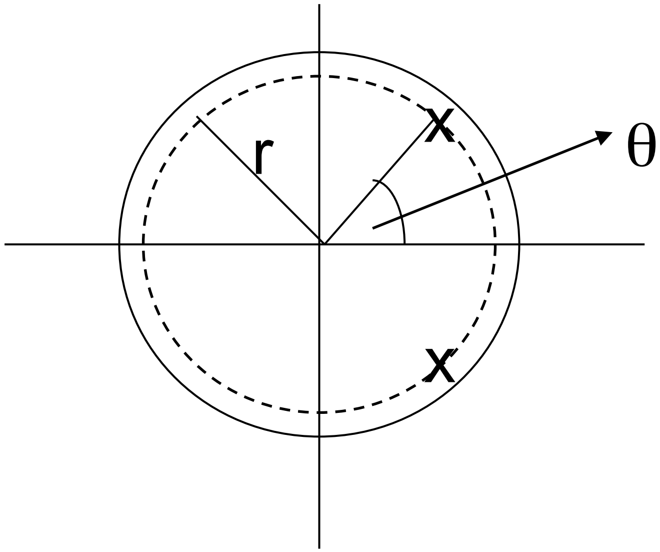
If $\alpha > 1$, h[n] grows without limit, $\Rightarrow$ H(Z) = unstable system.

Conclusion: A system is stable only if the poles (i.e., all poles) lie within the unit circle !

Clearly the same is true of signals: one or more poles outside the unit circle $\Rightarrow$ *unbounded signal*

The Z-Transform

Pole locations in Polar Coordinates



Two poles lying on a circle of radius 'r' making and angle θ w.r.t the Re(al) axis ⇒ So pole locations are r.[exp(jθ)] and r.[exp(-jθ)]

So $H(Z) = \dfrac{Y(Z)}{X(Z)} = \dfrac{1}{\left[Z - r.\exp(j\theta)\right]\left[Z - r.\exp(-j\theta)\right]}$

$$= \frac{1}{\left(Z^2 - 2rZCos\theta + r^2\right)}$$

The Z-Transform

Pole locations in Polar Coordinates

$$\Rightarrow Y(Z).\left(Z^2 - 2rZCos\theta + r^2\right) = X(Z)$$

$$\Rightarrow y[n] = 2rCos\theta y[n-1] - r^2 y[n-2] + x[n-2]$$

For a bounded (finite) output, r<1 ⇒ system is stable

Example Application - See figure 4.2 and equation 4.14

$$H(Z) = \frac{Z^2(Z-1)\left(Z^2+1\right)}{(Z+0.8)\left(Z^2+1.38593Z+0.9604\right)\left(Z^2-1.64545Z+0.9025\right)}$$

Two Re Zeros @ Z=0
One Re Zero @ Z=1
Two Im Zeros @ Z=±j

One Re Pole @ Z=-0.8

The Z-Transform

Pole locations in Polar Coordinates

Now compare;

$$\frac{1}{\left(Z^2 - 2rZCos\theta + r^2\right)} \quad \text{with} \quad \frac{1}{\left(Z^2 + 1.38593Z + 0.9604\right)}$$

$r^2 = 0.9604$ and $2rCos\theta = -\ 1.38593$

$\Rightarrow \qquad r = 0.98$ and $\theta = 45^O$

$$\frac{1}{\left(Z^2 - 1.64545Z + 0.9025\right)}$$

yields $r = 0.95$ and $\theta = 150^O$

Hence the system is stable because ALL $r < 1$!

The Z-Transform

Note:

(a) Comments on system stability apply only to poles and there is no dependence on zeros of the transform

(b) Zeros at the origin of the Z-plane produce only a time advance or delay but they have no other effect

Consider again: $H(Z) = \dfrac{1}{\left(Z^2 - 2rZCos\theta + r^2\right)}$

$$\Rightarrow y[n] = 2rCos\theta y[n-1] - r^2 y[n-2] + x[n-2]$$

So that y[0] = 0 and y[1] = 0

We can force y[n] to start @ n=0 by using (b) above !

The Z-Transform

Simply apply the time advance operator 'Z' twice:

$$H'(Z) = Z^2.H(Z)$$

$$\Rightarrow H'(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z^2}{\left(Z^2 - 2rZCos\theta + r^2\right)}$$

$$\Rightarrow \left(Z^2 - 2rZCos\theta + r^2\right)Y(Z) = Z^2 X(Z)$$

$$\Rightarrow y[n] = 2rCos\theta y[n-1] - r^2 y[n-2] + x[n]$$

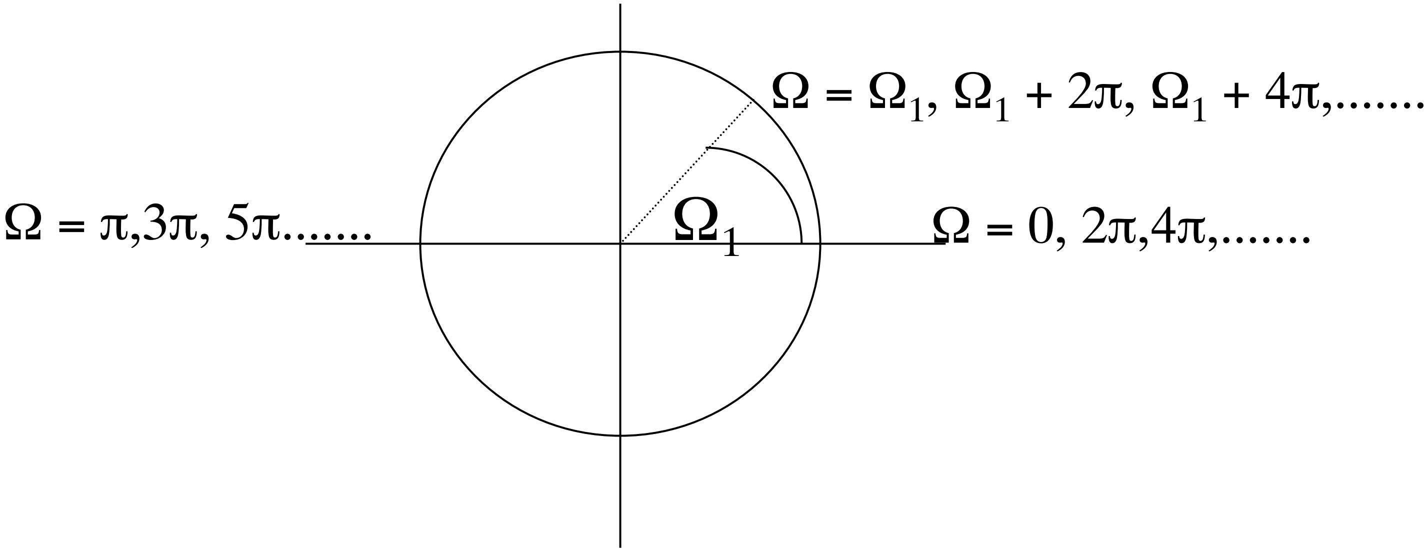
The Z-Transform

The Fourier Transform in the Z-Plane: Geometrical Methods

Setting Z = exp(-jΩ) we transform the Z-T into the the F-T

Where are the values of Z = exp(jΩ) in the Z plane ?

Magnitude r = 1; they lie on the unit circle



The Z-Transform

Ω_1 = 0 corresponds to the point Z = (1, j0)
on the Argand Plane.

As Ω increases, one moves anticlockwise
around the unit circle;
@ $\Omega = \pi$, Z = (-1, j0)

We know from Fourier Analysis that the FT repeats with a period of 2π. The Z-Transform (Z-plane) represention shows this automatically as any one point on the unit circle (1, Ω) can represents all frequencies $\Omega + 2n\pi$, n= 0, 1, 2, 3, 4, ...

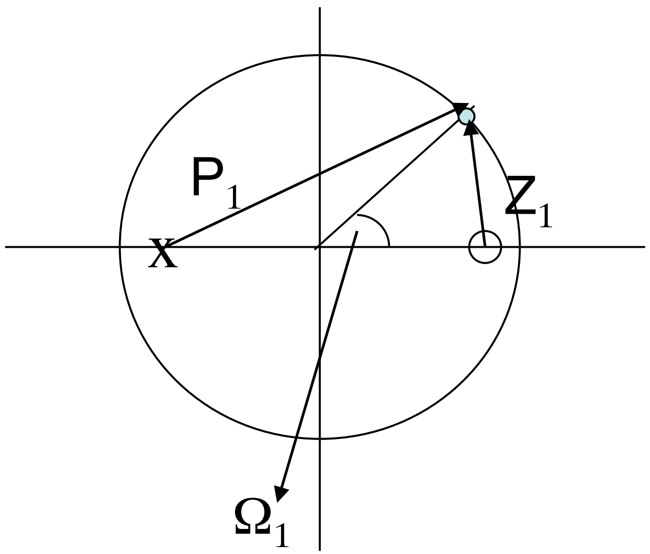
The Z-Transform

Consider the following LTI Processor with Z-Transform;

$$H(Z) = \frac{Z - 0.8}{Z + 0.8}$$

Setting Z = exp(jΩ)

$$\Rightarrow H(\Omega) = \frac{\exp(j\Omega) - 0.8}{\exp(j\Omega) + 0.8}$$



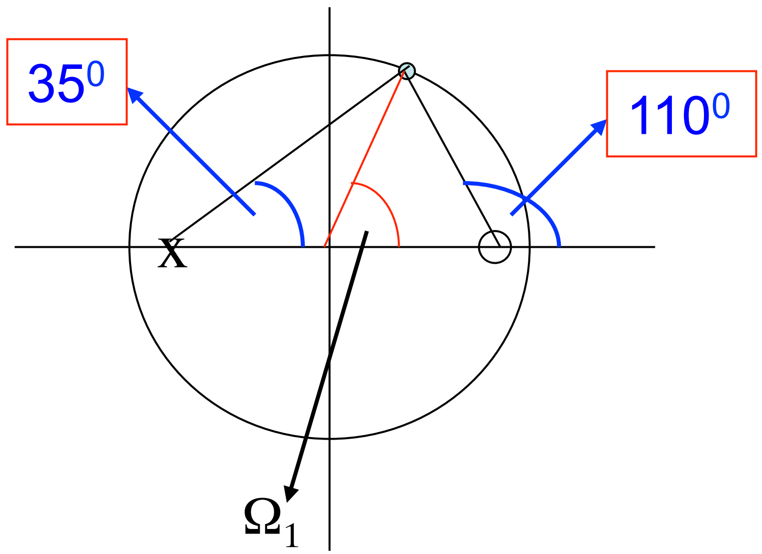
The Z-Transform

The numerator of H(Z) is determined by the magnitude (length) of the Zero Vector $|Z_1|$ and the denominator by the pole length $|P_1|$.

So the magnitude/gain of the frequency transfer function @ $\Omega = \Omega_1$

$$\left|H(\Omega)\right|_{\Omega_1} \propto \frac{|Z_1|}{|P_1|}$$

The phase shift or phase transfer function @ $\Omega = \Omega_1$



$$\left|\Phi_H(\Omega)\right|_{\Omega_1} = 110^0 - 35^0 = 75^0$$

The Z-Transform

By simple extension of this graphical method one can map the frequency transfer function of a simple processor

Ω	Z_1	P_1	$\|H(\Omega)\|$
0	0.2	1.8	0.11
$\pi/2$	1.28	1.28	1.00
π	1.8	0.2	9.00
2 π	0.2	1.8	0.11

From figure 4.6 one can see that as one travels around the unit circle (increasing frequency Ω), the gain $|H(\Omega)|$ goes through a maximum when $Z = \exp(j\Omega)$ passes near a pole, here e.g., @ $\Omega = \pi$.

The Z-Transform

More complex processor- consider again:

$$H(Z)=\frac{Z^2(Z-1)(Z^2+1)}{(Z+0.8)(Z^2+1.38593Z+0.9604)(Z^2-1.64545Z+0.9025)}$$

Zeros @ Z=0 (2nd order), Z =1 and Z = ±j
Poles @ Z = -0.8 {Real, Ω = 180^0 (π)}, a pole pair close to the unit circle @ Ω = ±45^0 ($\pi/4$) and a second pole pair at Ω = ±150^0 ($5\pi/6$) and also close to the unit circle.

Zeros: Expect 'true nulls' in the frequency response at Ω values specifying their location, here 0 and π

Poles: Expect peaks in the frequency response (gain) at Ω values specifying their location, here $\pi/4$, $5\pi/6$ and π

The Z-Transform

We can break more complex systems with many zeros and poles into simpler systems as follows:

$$|H(\Omega)| = K.\frac{|Z_1(\Omega)|.|Z_2(\Omega)|.|Z_3(\Omega)|........}{|P_1(\Omega)|.|P_2(\Omega)|.|P_3(\Omega)|........}$$

where K is a constant

$$\Phi_H(\Omega) = \left[\Phi_{Z_1}(\Omega) + \Phi_{Z_2}(\Omega) + \Phi_{Z_3}(\Omega).....\right] - \left[\Phi_{P_1}(\Omega) + \Phi_{P_2}(\Omega) + \Phi_{P_3}(\Omega).....\right]$$

$\Phi_{Z_n}(\Omega)$ - Phase vector of the n^{th} zero of H(Z)

$\Phi_{P_n}(\Omega)$ - Phase vector of the n^{th} pole of H(Z)

The Z-Transform

1^{st} and 2^{nd} order LTI Processors

Any LTI processor $H(Z) = K.\dfrac{(Z-Z_1)(Z-Z_2)(Z-Z_3)\dots\dots\dots}{(Z-P_1)(Z-P_2)(Z-P_3)\dots\dots\dots}$

may be represented by cascading a series of 1^{st} and 2^{nd} order LTI processor transfer functions

1^{st} order $\quad H_1(Z) = \dfrac{(Z-Z_1)}{(Z-P_1)} \quad$ Z_1 and $P_1 \in R$

2^{nd} order $\quad H_2(Z) = \dfrac{(Z-Z_2)(Z-Z_3)}{(Z-P_2)(Z-P_3)}$

$Z_2, Z_3, P_2, P_3 \in R$ or occur as complex conjugate pairs

The Z-Transform

1^{st} and 2^{nd} order LTI Processors - Frequency Selectivity

Poles placed at well defined Ω values (frequencies) and located near the unit circle will produce sharp peaks in the frequency response of a processor. An equal number of zeros placed at the Z-plane origin ensures that h[n] starts @ n=0

We write that a real pole for a 1^{st} order procesor is located at $Z = \alpha$

The complex conjugate pole pair for a 2^{nd} order processor are located at $Z = r.\exp(j\theta)$ and $Z = r.\exp(-j\theta)$

The Z-Transform

Hence one can write:

$$H_1(Z) = \frac{Z}{(Z-\alpha)}$$

$$H_2(Z) = \frac{Z^2}{\{Z - r.\exp(j\theta)\}\{Z - r.\exp(-j\theta)\}}$$

$$H(Z) = \frac{Z^2}{(Z^2 - 2rZCos\theta + r^2)}$$

The Z-Transform

1st order   Already looked at these in detail: (see Fig 4.9)

$0 < \alpha < 1 \Rightarrow$ Low pass filter

$-1 < \alpha < 0 \Rightarrow$ High pass filter

$$H_1(Z) = \frac{Z}{(Z-\alpha)} \quad , Z \rightarrow \exp(j\Omega) \quad \Rightarrow H_1(\Omega) = H_1(Z)\Big|_{Z=\exp(j\Omega)} = \frac{\exp(j\Omega)}{\exp(j\Omega)-\alpha}$$

$|H_1(\Omega)|$ has a maximum value (or gain) @ $\Omega = 0$

$$G_{\max} = \left|\frac{\exp(j0)}{\exp(j0)-\alpha}\right| = \frac{1}{1-\alpha}$$

$|H_1(\Omega)|$ has a minimum value (or gain) @ $\Omega = \pi$

$$G_{\min} = \left|\frac{\exp(j\pi)}{\exp(j\pi)-\alpha}\right| = \frac{1}{1+\alpha}$$

The Z-Transform

1^{st} order cont'd

So the closer α is to 1 (i.e., the closer the pole is to the unit circle), the larger the maximium gain (G_{max}), the smaller the minimum gain (G_{min}) and so the narrower the bandwidth of the filter.

In summary, as one moves the pole of a 1st order filter closer to the unit circle:

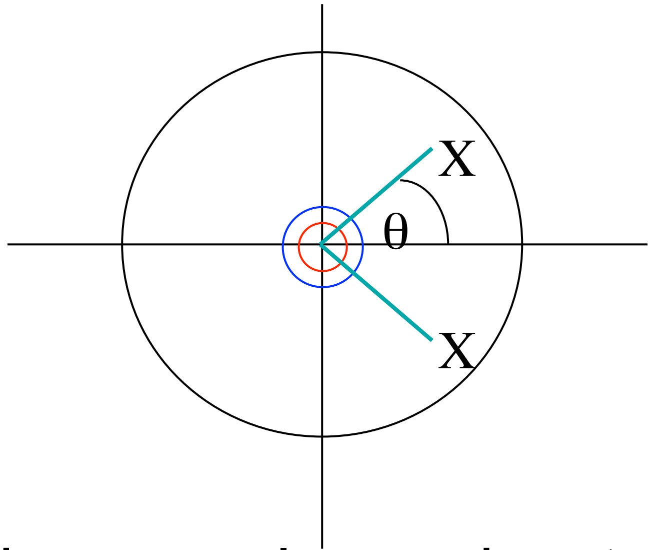
(i) The peak gain increases
(ii) The bandwidth decreases
(iii) The impulse response ($1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots$) slows

(ii) and (iii) $\Rightarrow$ Narrow the bandwidth and slow the processor

The Z-Transform

2nd order filters

$$H_2(Z) = \frac{Z^2}{\left(Z^2 - 2rZ\cos\theta + r^2\right)}$$



The processor has complex conjugate pole pairs
@ Z=r.exp(jθ) and a 2nd order zero at the origin (Z=0)

θ determines the centre frequency of the processor /signal and 'r' determines the 'selectivity' or bandwidth

The Z-Transform

2nd order filters

$$H_2(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{\left(1 - 2rCos\theta Z^{-1} + r^2 Z^{-2}\right)}$$

$$\Rightarrow \; y[n] = 2rCos\theta y[n-1] - r^2 y[n-2] + x[n]$$

Letting Z = exp(jΩ) ⇒

$$H_2(\Omega) = \frac{1}{\left(1 - 2rCos\theta \exp(-j\Omega) + r^2 \exp(-j2\Omega)\right)}$$

Hence:

$$\left|H_2(\Omega)\right| = \frac{1}{\left[\left(1 - 2rCos\theta Cos\Omega + r^2 Cos2\Omega\right)^2 + \left(2rCos\theta Sin\Omega - r^2 Sin2\Omega\right)^2\right]^{1/2}}$$

The Z-Transform

Maximum gain occurs at the centre frequency, $\Omega = \theta$.

$$|H_2(\theta)| = G_{max} = \frac{1}{\left[\left(1 - 2rCos^2\theta + r^2Cos2\theta\right)^2 + \left(2rCos\theta Sin\theta - r^2Sin2\theta\right)^2\right]^{1/2}}$$

Let A= $\left(1 - 2rCos^2\theta + r^2Cos2\theta\right)$

Let B= $\left(2rCos\theta Sin\theta - r^2Sin2\theta\right)$

$$\Rightarrow G_{max} = \frac{1}{\sqrt{A^2 + B^2}}$$

Program 11 used to study the effect of r, θ for 2nd O systems using the expressions above - very powerful - write your own

See Figure 4.11 for typical results

The Z-Transform

Figure 4.11 examples

(a) Cascaded 1st order LPFs: $r = 0.9$, max gain @ $\Omega = \theta = 0^0$ (dc) !

(b) BPF centred @ $\Omega=\theta=25^0$ (~15 Samp/cyc.), $r=0.99$ (narrow passband)

(c) BPF with wide bandpass ($r = 0.8$) centred @ $\Omega=\theta=110^0$ (~3 Samp/cyc.)

(d) Cacaded 1st order HPFs: $r=0.9$, max gain @ $\Omega=\theta=180^0$ (2 Samp/cyc.)

Note: When the pole 'r' value is close to '1', you should also expect rapid variations in the phase transfer function $\Phi_H(\Omega)$ as Ω sweeps through the design frequency 'θ' !

Reminder: In sampled data systems we specify 'frequency' in samples/cycles or equivalently 'radians' or 'degrees' !

The Z-Transform

Effect of initial conditions on the Z - Transform

Useful in two cases:

(a) The system may not have settled following the application of a prior input

(b) The input signal was applied prior to n = 0 and it is required to assess its subsequent effects on the output

Consider the following:

If: $x[n] \rightarrow X(Z)$

Then: $x[n - n_0] \rightarrow X(Z).Z^{-n_0}$

$\boxed{\text{We can in fact write: } x[n - n_0].u[n - n_0] \rightarrow X(Z).Z^{-n}}$

Multiplication by $u[n - n_0]$ ensures that $x[n]$ is zero for $n<n_0$

The Z-Transform

Suppose that we define $x_1[n]$ as a signal identical to $x[n]$ but not necessarily starting @ n = 0

i.e., $x_1[n] \rightarrow x[n-1]$

Then: $X_1(Z) = \sum_{n=0}^{n=\infty} x[n-1].Z^{-n} = x[-1] + \sum_{n=1}^{n=\infty} x[n-1].Z^{-n}$

But: $\sum_{n=1}^{n=\infty} x[n-1]Z^{-n} = Z^{-1}\sum_{n=1}^{n=\infty} x[n-1]Z^{-(n-1)}$

And also: $Z^{-1}\sum_{n=1}^{n=\infty} x[n-1]Z^{-(n-1)} = Z^{-1}\sum_{n=0}^{n=\infty} x[n]Z^{-(n)}$

So $X_1(Z) = x[-1] + Z^{-1}\left\{\sum_{n=0}^{n=\infty} x[n]Z^{-n}\right\} = x[-1] + Z^{-1}X(Z)$

The Z-Transform

Similarly it can be shown that:

If: $x_2[n] = x[n - 2]$

Then: $X_2(Z) - x[-2] + x[-1]Z^{-1} + Z^{-2}X(Z)$

Example - Consider: $y[n] = \alpha . y[n-1] + x[n]$

We can rewrite as: $y[n] - \alpha . y[n-1] = x[n]$

Taking ZTs: $Y(Z) - \alpha\left\{ y[-1] + Z^{-1} Y(Z) \right\} = X(Z)$

$$\Rightarrow Y(Z)\left\{1 - \alpha . Z^{-1}\right\} = X(Z) + \alpha . y[-1]$$

$$\Rightarrow Y(Z) = \frac{X(Z) + \alpha . y[-1]}{\left(1 - \alpha . Z^{-1}\right)}$$

The Z-Transform

Case in point:

$$y[-1] = -\frac{1}{\alpha} \quad \Bigg| \quad x[n] = \delta[n] \quad \Bigg| \Rightarrow X(Z) = 1$$

We have that:

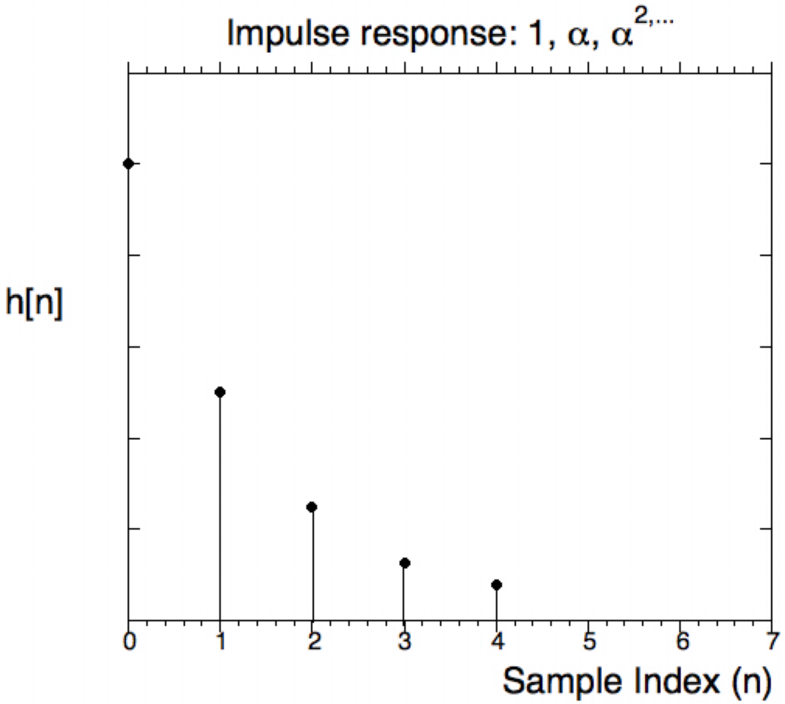
$$\Rightarrow Y(Z) = \frac{X(Z) + \alpha . y[-1]}{\left(1 - \alpha . Z^{-1}\right)}$$

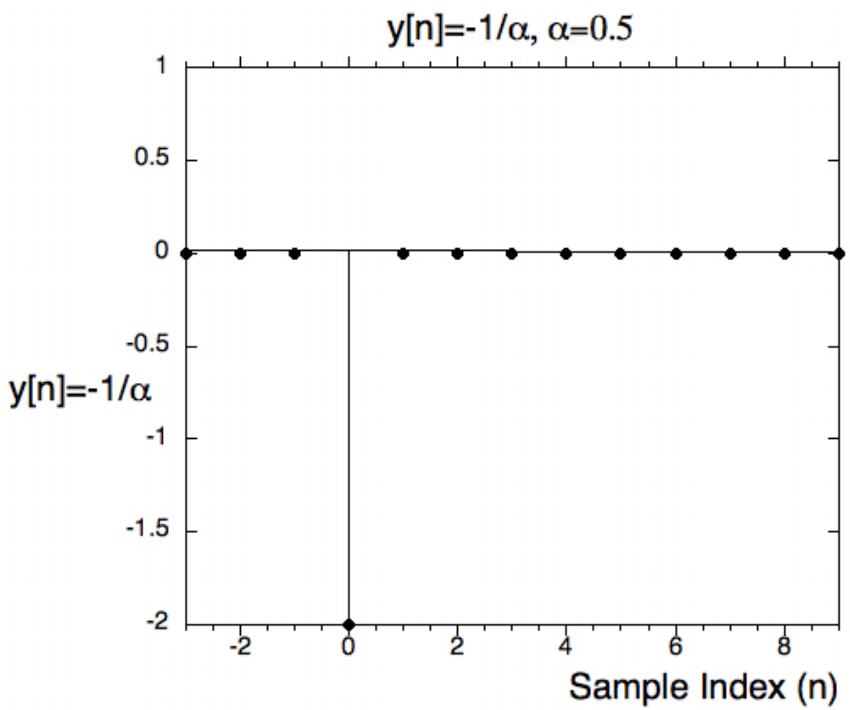
$$\Rightarrow Y(Z) = \frac{X(Z) + \alpha\left(-\frac{1}{\alpha}\right)}{\left(1 - \alpha Z^{-1}\right)} = 0$$

i.e., y[n] = 0 for all n> 0 !!

What has happened to the signal ? - we had a system with impulse response h[n] = 1, α, α^2, α^3, α^4,.....

The Z-Transform





So simply having an initial (prior) condition like $y[n]=-1/\alpha$, the impulse response can be set to zero ! The initial (prior) state of the system exactly cancels the impulse response so that the filter output power becomes identically zero !!

The Z-Transform

So importantly, h[n] is the impulse response of a processor only if all initial conditions are zero !

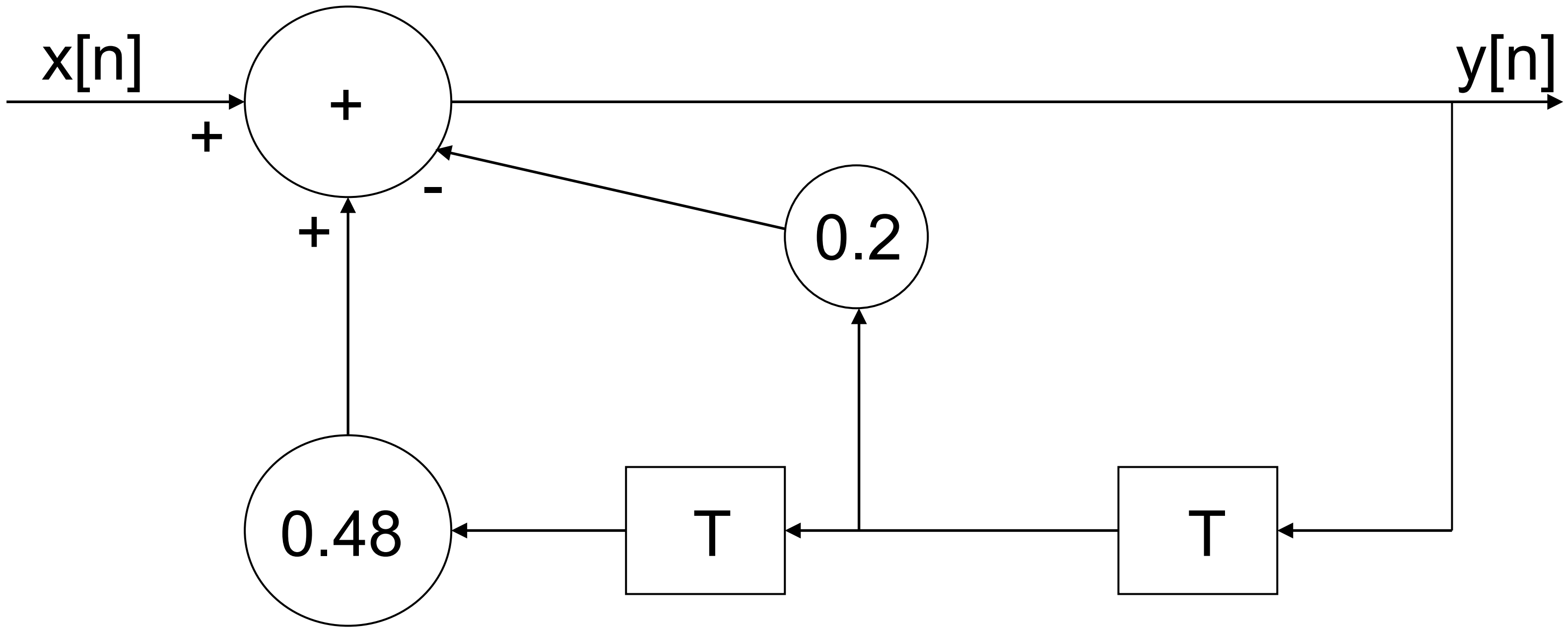
i.e., H(Z) = Y(Z)/X(Z) ONLY under zero initial conditions

Problem (for you):

Using $y[n]-\alpha.y[n-1]=x[n]$ compute h[n] for x[n] = δ[n] = 1 and y[-1] = 1/α

The Z-Transform

Example



From the above block diagram we have that:
$y[n] = -0.2y[n-1] + 0.48y[n-2] + x[n]$

The Z-Transform

Find y[n] when x[n] = δ[n] for

(i) Zero initial conditions

(ii) y[-1] = -1.25 and y[-2] = -0.52083

Taking ZTs of both sides we get for non-zero initial conditions:

$$Y(Z) = \frac{X(Z) - 0.2y[-1] + 0.48y[-2] + 0.48y[-1]Z^{-1}}{1 + 0.2Z^{-1} - 0.48Z^{-2}}$$

Case (i): Zero initial conditions

x[n] = δ[n], ⇒ X(Z) = 1 and y[n] = h[n] = 0, for all y[n] (n<0).

$$\Rightarrow Y(Z) = \frac{1}{\left(1 + 0.2Z^{-1} - 0.48Z^{-2}\right)} = \frac{Z^2}{\left(Z^2 + 0.2Z - 0.48\right)}$$

The Z-Transform

$$\Rightarrow Y(Z) = \frac{Z^2}{(Z+0.8)(Z-0.6)} = \frac{AZ}{(Z+0.8)} + \frac{BZ}{(Z-0.6)}$$

Multiply both sides by (Z + 0.8)(Z - 0.6). Values of A & B must be true for all Z. Try poles of Y(Z), i.e., Z = -0.8 & 0.6.

$$\Rightarrow Z^2 = AZ(Z-0.6) + BZ(Z+0.8)$$

Z = 0.6, $\Rightarrow$ B(0.6)(1.4) = 0.36, $\Rightarrow$ B = 0.4286

Z = -0.8, $\Rightarrow$ A(-0.8)(-1.4) = 0.48, $\Rightarrow$ A = 0.5714

Hence $\Rightarrow Y(Z) = \dfrac{0.5714.Z}{(Z+0.8)} + \dfrac{0.4286Z}{(Z-0.6)}$

The Z-Transform

Using table 4.1 on Inverse Z-Transforms we have that:

$$\frac{Z}{Z-a} \rightarrow a^{n}.u[n]$$

$$\Rightarrow y[n] = 0.5714.(-0.8)^{n} u[n] + 0.4286.(0.6)^{n} u[n]$$

See figure 4.13(b)

The Z-Transform

(ii) Non-zero initial conditions: y[-1] = -1.25 & y[-2] = -0.52083

$$Y(Z) = \frac{X(Z) - 0.2y[-1] + 0.48y[-2] + 0.48y[-1]Z^{-1}}{1 + 0.2Z^{-1} - 0.48Z^{-2}}$$

$$Y(Z) = \frac{X(Z) - 0.2[-1.25] + 0.48[-0.52083] + 0.48[-1.25]Z^{-1}}{\left(1 + 0.8Z^{-1}\right)\left(1 - 0.6Z^{-1}\right)}$$

$$Y(Z) = \frac{X(Z) + 0.25 - 0.25 - 0.6Z^{-1}}{\left(1 + 0.8Z^{-1}\right)\left(1 - 0.6Z^{-1}\right)}$$

$$\Rightarrow Y(Z) = \frac{1}{\left(1 + 0.8Z^{-1}\right)} = \frac{Z}{Z + 0.8}$$

$$\boxed{\Rightarrow y[n] = (-0.8)^n u[n]}$$

See figure 4.13(c)