Unit 57 Nonlinear circuits and chaos

- Chaotic systems are characterized by a very sensitive dependence on the initial conditions which causes initially adjacent system states to diverge exponentially.
- A chaotic system must have at least three degrees of freedom if the system state is never to repeat.
- Chaotic systems may start in a stable state but become periodic and then chaotic as some parameter is varied by following a route to chaos. A common route to chaos is associated with period doubling.
- Many chaotic systems enter a stable, nonrepeating chaotic orbit which when projected onto a plane forms a pattern called a strange attractor.

We have examined linear systems in some detail since such linear systems are the most commonly used in electronics. Even when the system becomes mildly nonlinear, such as is shown in the example of saturation illustrated in Figure 53.3, the system still remains stable and predictable. What we will examine in this unit are the characteristics of nonlinear circuits which become chaotic but calculable and are described under the heading of deterministic chaos.

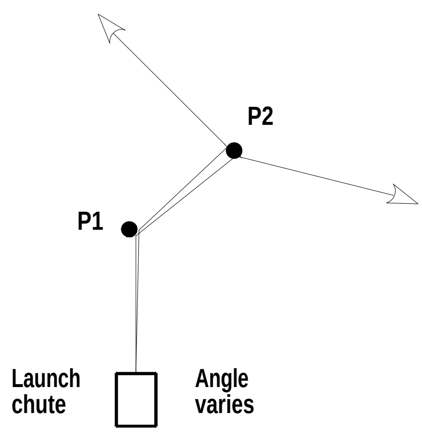

Figure 57.1: Pinball machine.

The concept of a deterministic chaotic system whose eventual state exhibits a sensitive dependence of initial conditions is well illustrated by a simple pinball mechanical system as shown in Figure 57.1.

A ball is launched at the array of pegs. It strikes the first and recoils at an angle which depends strongly on the impact direction. It then impacts on a second pin again having an angle of recoil depending on the line of incidence. The final position of the ball is plotted after a fixed time. As time increases the position of the ball becomes more and more sensitive to the initial trajectory of the ball. The final position in a chaotic system becomes an exponentially diverging function of the number of recoils off the pegs. Or to put it differently, two initially adjacent trajectories will diverge to give an eventual position which could be anywhere in the available space.

In the system in Figure 57.1, we can use only one peg and plot the position of impact of the ball on the border as a function of the angle of launch. The distribution of impact points is calculable. Put in a second peg and repeat the calculation or the experiment. Some of the trajectories such as those where the ball does not strike more than one peg are easily calculated. As the number of pegs is increased the distribution of impact points becomes more and more random or chaotic.

This example has illustrated one of the properties of chaotic systems, an exponential divergence characterized by what is called a Lyanopov exponent.

Another feature of some chaotic systems is that the systems are initially stable but as some parameter is varied they become chaotic. There is what is called a route to chaos, the most common of which is the period doubling route to chaos.

Consider a population of frogs living in a pond. In the simplest model, the population x_{n+1} in any one year depends on the population x_n in the previous year and the reproduction rate, r, so that:

$$x_{n+1} = r \times x_n$$

It is easily seen that the frog population is only stable and constant when $r = 1$. If $r > 1$ there is a population explosion and if $r < 1$ the frogs become extinct.

However, frogs eat flies and if there are more frogs there will be less flies, so we can visualize an improved model of the population dynamics in which we describe the number of flies available for food by a term $(1 - x_n)$ where $x_n = 0$ corresponds to no frogs and $x_n = 1$ corresponds to the pond full of frogs! Now let us change the population growth model and assume that the number of frogs in the next generation depends on the reproduction rate, r, the number of frogs, x_n, and the available food (flies), described by $(1 - x_n)$.

The number of frogs in the next generation is then given by the logistic equation:

$$x_{n+1} = r \times x_n \times (1 - x_n)$$

In order to examine the behaviour of the population we carry out a numerical model of the system. This is an operation which is very frequently performed in modelling chaotic systems. In dynamical systems it involves numerical modelling of the differential equations which describe the system and frequently requires significant computing power. In this case, however, we will use a straightforward iteration to calculate a population from the parameters and the previous year's population.

The logistic equation $x_{n+1} = r \times x_n \times (1 - x_n)$ should be programmed into a computer and run for about 1000 iterations for various values of the parameter r. When the last 20 values of x_n are printed to the screen and are are examined it is found that for any initial nonzero value of x_0 and for a range of values for r, number sequences similar to those shown in the table will obtained.

$r =$	1.0	1.5	2.0	2.5	3.0	3.5	3.55
1001	0	0.33	0.5	0.6	0.674	0.501	0.540
1002	0	0.33	0.5	0.6	0.659	0.874	0.882
1003	0	0.33	0.5	0.6	0.674	0.383	0.370
1004	0	0.33	0.5	0.6	0.659	0.826	0.827
1005	0	0.33	0.5	0.6	0.674	0.501	0.506
1006	0	0.33	0.5	0.6	0.659	0.874	0.887
1007	0	0.33	0.5	0.6	0.674	0.383	0.354
1008	0	0.33	0.5	0.6	0.659	0.826	0.812
1009	0	0.33	0.5	0.6	0.674	0.501	0.540
1010	0	0.33	0.5	0.6	0.659	0.874	0.882

This program is run for values of r extending from 0 to 4 and the resulting values of x are plotted on a graph as shown in Figure 57.2. From the table there is only one value, $x = 0.6$, for $r = 2.5$ but there are four values of x for $r = 3.5$. At each bifurcation point on the graph the number of iterations which are required before the pattern repeats increases by a factor of 2 which is why this is named the 'Period doubling route to chaos'. For values of r greater than 4 the pattern does not repeat and the succession of numbers is termed chaotic. This sequence of numbers for $r = 4$ is not a set of random numbers, however, because each number is computed or determined from the previous number and the system is therefore said to exhibit deterministic chaos.

So we now have a system which may be stable for a certain range of a parameter but which then becomes more and more unstable along a period

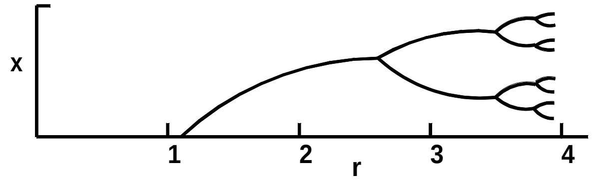

Figure 57.2: Period doubling route to chaos.

doubling route as the parameter is varied away from the stable region. An important distinction should be made at this point. In an electronic system distortion is characterized by higher Fourier harmonics appearing in the output but this is not chaotic behaviour. Period doubling is characterized by subharmonics appearing in the output and this is a signature of the imminence of chaos.

Linear systems do not exhibit chaotic behaviour. Nonlinear systems usually introduce higher frequency Fourier components and occasionally exhibit chaotic behaviour but, until recently, systems which exhibit chaotic behaviour were either not understood or not recognized and therefore tended not to be discussed in textbooks. Any known but unrecognized cases of chaotic behaviour were regarded as bad engineering and avoided. Now that we can recognize, describe and understand the phenomenon of chaos there is a trend towards utilizing chaotic systems for useful purposes.

A number of electronic circuits have been proposed as examples of circuits which exhibit chaotic behaviour but in some cases, dating from the early days of the subject, electronic reality did not correspond to the mathematical models which purported to show and explain the chaotic behaviour.

We will restrict our discussion of chaotic circuits to one of the better characterized chaotic circuits which was initially developed by Leon Chua and which is now named after him.

In this treatment of chaotic electronic systems we take a simple system and attempt to follow through the analysis of the system from the viewpoint of electronics and not from the more abstract mathematical viewpoint. This approach to understanding the simpler case should then be applicable, by extension, to more complex chaotic electronic systems. Chaotic systems are nonlinear and the usual approach of using differential equations in the analysis will not work so numerical methods are employed. This leads to a loss of intuitive understanding of the circuits owing to the intervening layer of programming and modelling.

In carrying out the analysis, the principle of Occam's Razor will be applied. While the principle is ascribed to William of Occam, it does not actually appear stated in his works. It is that 'Entities are not to be multiplied without necessity' or in more modern phrasing 'Keep it simple'. Unfortunately owing to the complex nature of chaotic systems, the explanations are not always simple but we will endeavour to keep the flow of the argument as straightforward as possible.

The fundamental unit used in this circuit is a device called a Chua diode. It is essentially a nonlinear negative resistance. A passive negative resistance is impossible since a negative resistance is essentially a device which outputs electrical power, so there must be an energy source somewhere in the system. In our case the energy source is the power supply for the op-amp. However, we have already met a negative resistance when we discussed the photodiode. The lower right hand quadrant of the I–V characteristic for a photodiode shown again in Figure 57.3 is a region where the voltage across the device is positive but the current is negative thus giving a negative resistance. The light energy falling on the photodiode is being converted into electrical energy and is driving the external circuit. By a similar argument a battery or a dynamo can also be considered as a negative resistance device.

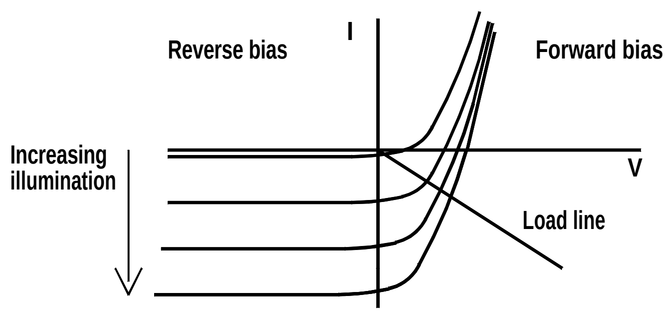

Figure 57.3: Photodiode characteristic.

A circuit which synthesizes a Chua diode is shown in Figure 57.4. We start with two diodes having resistors in series with them as shown in Figure 57.4 (a). The diodes do not conduct until the knee voltage of 0.7 V is reached and after that, the current is limited by the 3.3 kΩ series resistor so the I–V characteristic is as shown.

The second part of the circuit is shown in Figure 57.4 (b). In analyzing the operation of this circuit we use the rule that the voltage difference between the input terminals of the op-amp is zero. The voltage across the 1.1 kΩ resistor is therefore equal to the voltage at the top of the circuit. The output voltage from the op-amp must then be sufficient to drive current through the 300 Ω and 1.1 kΩ in series to give this voltage. If a voltage of 2 V is present at the top of the circuit then the op-amp output is $\frac{1100+300}{1100} \times 2 = 2.54\,\text{V}$. This op-amp output is also applied to the upper 300 Ω resistor and gives a current of $\frac{2-2.54}{300} = -1.8\,\text{mA}$ in the input corresponding to an input resistance of $-1.1\,\text{k}\Omega$.

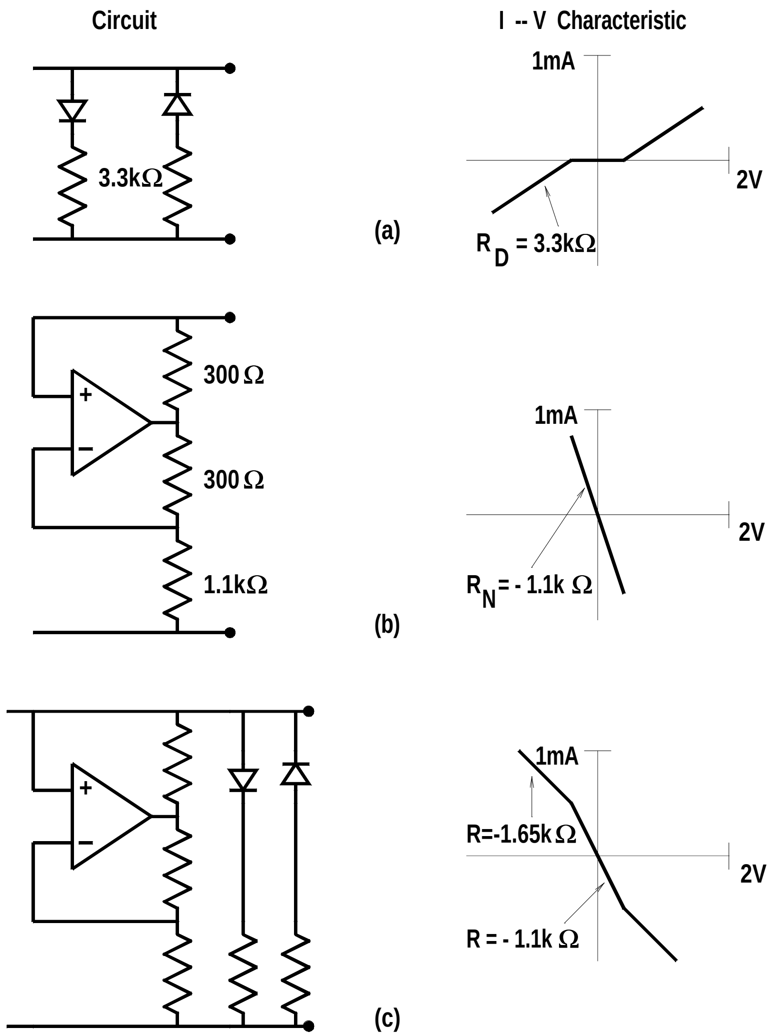

Figure 57.4: The operation of Chua's diode.

When these two circuits are placed in parallel as shown in Figure 57.4 (c) the central region has a resistance of $-1.1\,\text{k}\Omega$ but the outer regions have a resistance of $-1.65\,\text{k}\Omega$ corresponding to $+3.3\,\text{k}\Omega$ in parallel with $-1.1\,\text{k}\Omega$.

In combining the characteristics for each of the circuit segments it should be noted that since the two circuits are in parallel, we must add the currents at each voltage to get a resultant. When you are experimenting with this circuit, you can observe these characteristics by using a curve tracer which applies an alternating drive voltage to a circuit and measures the resulting current. The voltage is displayed on an oscilloscope X axis and the current is displayed on the Y axis as in the I–V characteristics. Some oscilloscopes have a component test facility which does the same thing but with less control over the voltages which can be applied. Alternatively you could use a circuit similar to that in Figure 17.4 where a function generator is used to drive the circuit (in place of R) and the capacitor is replaced by a low valued resistor (10 Ω) which is used to sense the current. The oscilloscope is then used in XY mode.

Now put a variable resistance and a parallel LC in series with this Chua diode circuit as shown in Figure 57.5. The variable resistance should be adjusted until oscillation is obtained. If the resistance of the variable resistor is then measured (out of circuit) it will be found that the value which gives stable oscillation is between the resistance of the two segments of the Chua diode (1.1 kΩ and 1.65 kΩ in this example). The oscillation is driven by the negative damping effect of the negative resistance of the Chua diode. The amplitude does not increase indefinitely because the series resistance of the Chua diode and the R_V becomes positive for larger amplitudes of oscillation thus limiting the amplitude.

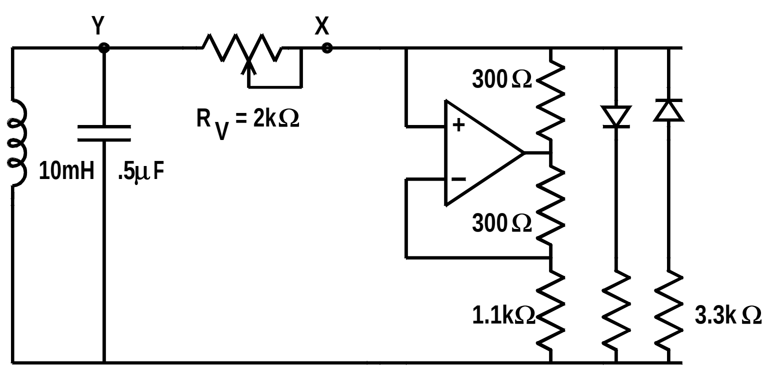

Figure 57.5: Chua's oscillator circuit.

This circuit oscillates with an approximately sinusoidal waveform at a frequency given by $f = \frac{1}{2\pi\sqrt{LC}}$ as shown in the oscilloscope tracing in Figure 57.6 of the waveforms at points X and Y.

If a capacitor is added to the circuit between point X and ground to give the circuit shown in Figure 57.7, then a circuit is formed which exhibits chaotic oscillation. In the circuit in Figure 57.7 we have replaced the op-amp and two diodes forming the Chua diode by the boxed resistor symbol which is now the more usual representation of the Chua diode. This is appropriate

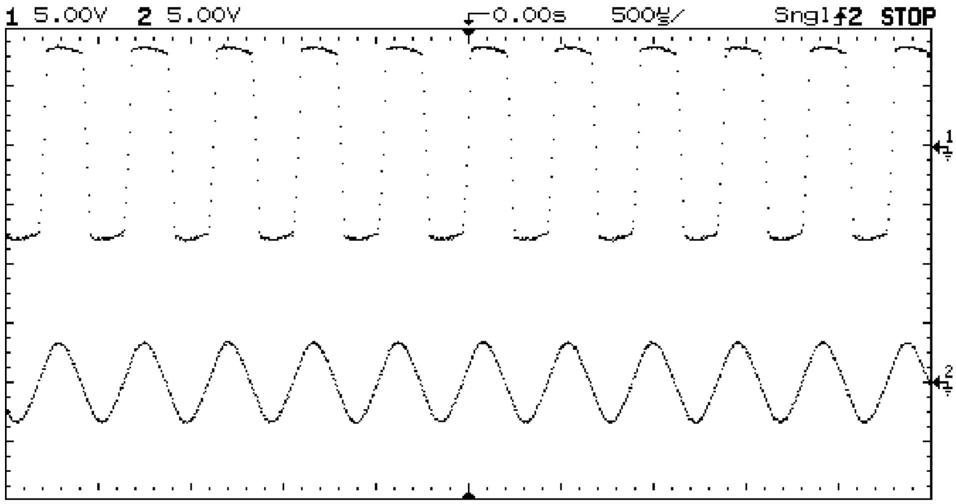

Figure 57.6: Storage oscilloscope printout of Chua oscillator waveforms.

since the nonlinear negative resistance diode is now available as an integrated circuit component.

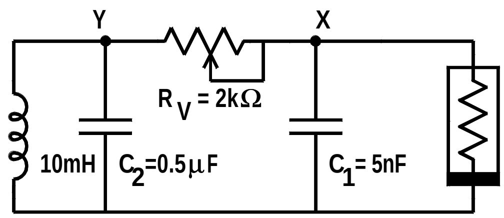

Figure 57.7: Chua's chaotic oscillator.

After the variable resistor R_V has been adjusted with some delicacy you should get the oscilloscope waveforms similar to those shown in Figure 57.8 with the signals being taken from points X and Y in the circuit. If you fail to obtain these waveforms then you may have to substitute slightly different values for some of the components. It has been found that the inductor is a critical component. If the resistance of the inductor is too large the circuit will not operate. Also you might try substituting slightly different values for the 1.1 kΩ shown in Figure 57.5. The difficulty is that the operation of this circuit depends on differences of about 1% between two resistors which are only specified to a 5% tolerance. One solution is to use a variable resistor instead of the 1.1 kΩ resistor and trim the circuit until it operates.

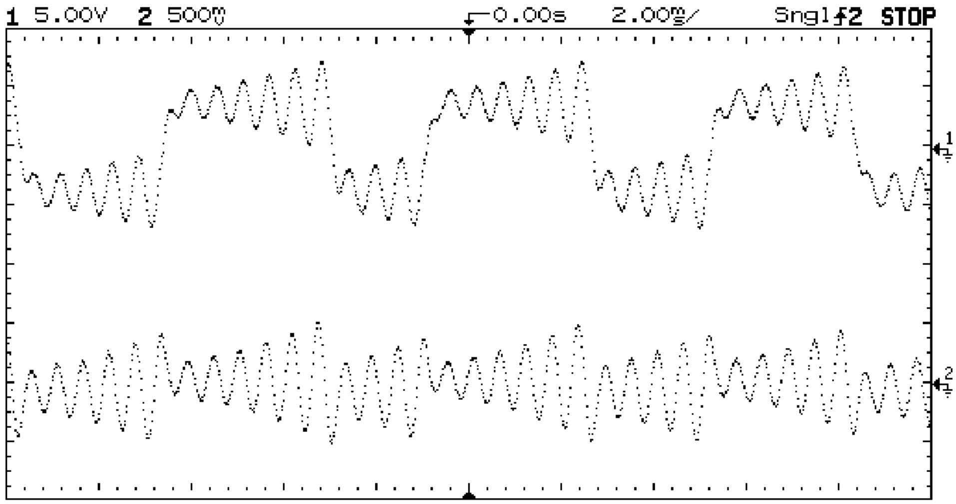

Figure 57.8: Voltage waveforms measured at points X and Y of Figure 57.7.

Another form of presentation of the performance of a chaotic oscillator circuit is shown in Figure 57.9 where the oscilloscope is operated in XY mode with the same signals as those in Figure 57.8. This representation shows what is called the double scroll chaotic attractor for the circuit.

We will now present an explanation in electronic terms of the operation of the chaotic oscillator circuit. A full mathematical treatment of the operation is available in the literature (see, for instance, Kennedy, M.P., *IEEE Transactions on Circuits and Systems*, **40**, (10), 640, 1993) but the mathematical model is not necessarily the best way of obtaining a first understanding of the operation of the circuit.

This explanation is presented essentially as a superposition of three distinct mechanisms which operate in the circuit.

First consider a potential divider of R_V and R_N where R_N is the Chua diode nonlinear negative resistance as shown in Figure 57.4 (c).

If a voltage is applied to this potential divider, as in Figure 57.10, the output voltage will be:

$$V_{out} = \frac{R_N}{R_V + R_N} \times V_{in}$$

But R_N is negative so when R_V is slightly less than the magnitude of R_N we will obtain an output voltage from the potential divider which is **larger** than the input voltage. The voltage at point Y across the LC in Figure 57.7 and shown in the lower trace in Figure 57.8 can be considered as the input to the potential divider and the voltage at point X in Figure 57.7, shown as the

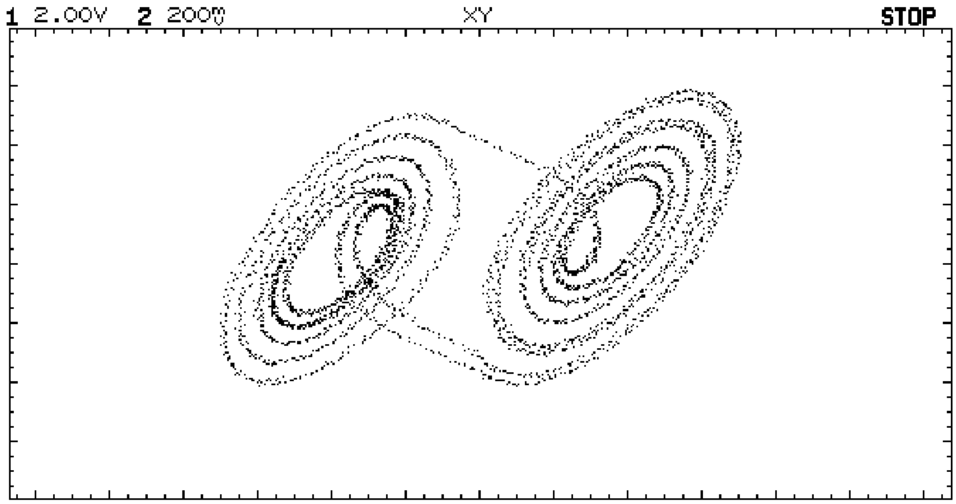

Figure 57.9: Double scroll chaotic strange attractor.

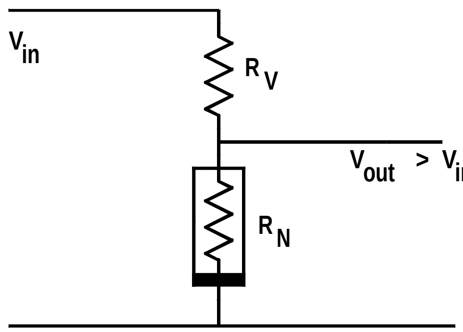

Figure 57.10: Negative resistance amplifier.

upper trace in Figure 57.8, can be considered as the output. Examination of the two traces in Figure 57.8 does show that there is an amplification and that the output of the potential divider is indeed greater than the input. The sensitivity settings of the oscilloscope are shown at the top left of the display.

Next, consider the series circuit of the R_V and the R_N of the Chua diode. Since these are in series we combine the I–V characteristic curves by adding the voltages horizontally for each current to give the resulting characteristic as shown in Figure 57.11.

This characteristic has an unstable equilibrium point at the origin and if an LC is connected across the input, the negative resistance of the I–V characteristic causes oscillations to grow in the LC owing to the negative damping of the characteristic. This then gives a zero DC bias and an exponentially

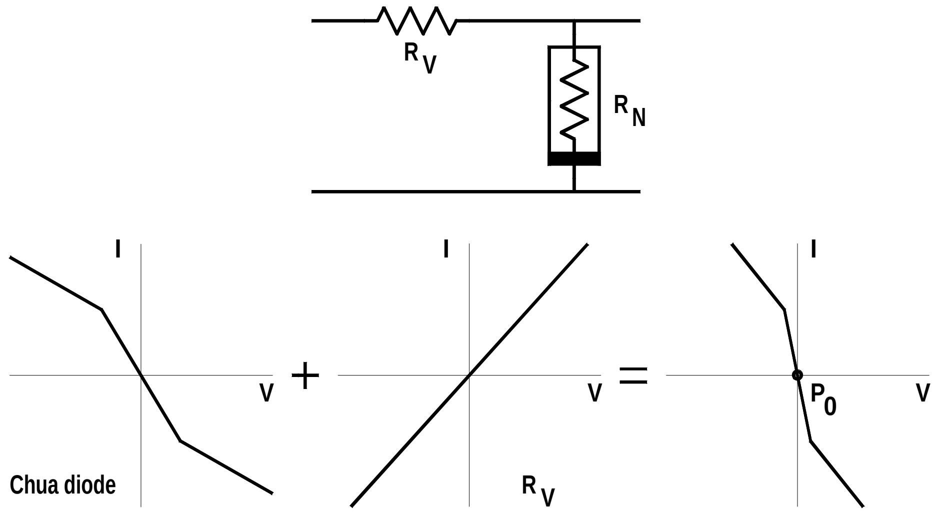

Figure 57.11: I–V characteristics for series R_V and R_N.

growing oscillation as shown in the lower trace of Figure 57.8.

Thirdly we consider the voltage across C_1 at point X in the circuit as shown in Figure 57.12. In this case we analyze in terms of the R_V and R_N in parallel and we get the resultant characteristic as shown in Figure 57.12 by adding the currents vertically for each voltage.

This characteristic gives two stable operating points at P_1 and P_2 with an unstable region in between at the origin.

When these three mechanisms are superimposed we see that the voltage across this parallel combination is driven by the amplified output oscillation of the potential divider and that the circuit jumps from one operating point to the other (P_1 to P_2 or P_2 to P_1) as soon as the voltage at X crosses the corners of the characteristic. The energy which has been built up in the oscillations in the LC is transferred to the storage capacitor C_1 and the oscillations in the LC have to start to grow from zero again. This mechanism gives the steps in the waveform across the capacitor C_1 as shown in the upper trace in Figure 57.8 which also has the amplified exponentially growing oscillations superimposed on the steps.

It can also be seen that the value of the capacitance, C_1, is critical. It must be large enough to store the energy built up in the oscillation in the LC circuit but it must not be so large as to smooth totally the output oscillation across C_1 and the potential divider. A value of C_1 such that $9C_1 = C_2$ usually gives satisfactory operation but you should try other values.

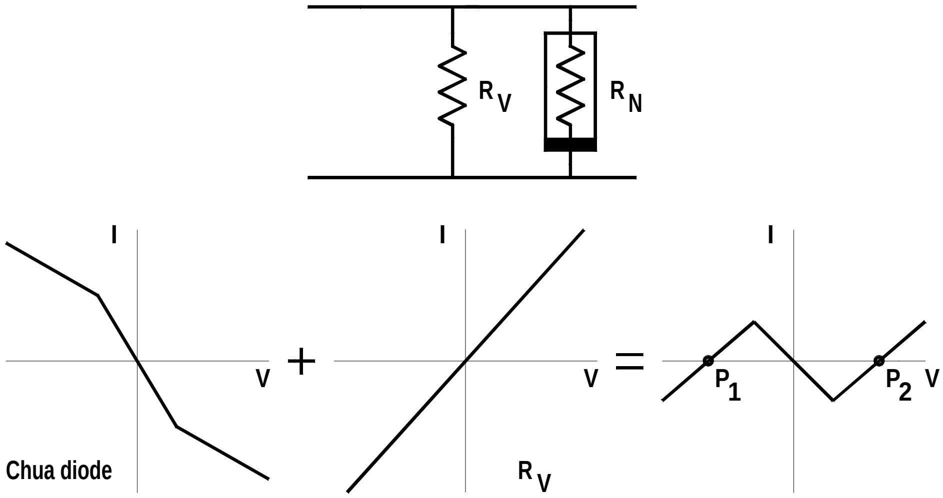

Figure 57.12: I–V characteristics for parallel R_V and R_N.

In the discussion of the logistic function we showed that the equation $x_{n+1} = r \times x_n \times (1 - x_n)$ shows a period doubling route to chaos as the parameter r is increased. If the R_V in the Chua circuit is varied slowly and if the circuit is biased towards one of the basins of oscillation by using, say, 5.6 kΩ and 3.3 kΩ resistors in series with the diodes in Figure 57.4 (a) then a period doubling sequence for the oscillations can be observed and is manifested by a progression from a single closed loop to a double loop to a four fold closed loop when the display is set to XY mode as in Figure 57.9.

In the sinusoidal oscillators which were described in Unit 53, the amplitude was stabilized by the use of a nonlinear feedback component such as a miniature bulb or a thermistor. This gave a stable loop gain of 1 and an oscillator which operated in the linear region. This can be compared with the controlled or limited amplitude oscillations which are obtained with the Chua circuit. In the case of the Chua oscillator, Figures 57.5 and 57.6, the amplitude stabilizes and the value of the voltage at some future time is calculable from $V = V_0 \sin(2\pi f t + \phi)$. In the case of the Chua chaotic oscillator, the loop gain is greater than 1 but the oscillation is prevented from growing indefinitely by the switching action in the circuit which kills a large oscillation and restarts the oscillation. The result is that the voltage at some future time is calculable and deterministic but is not predictable owing to the sensitivity of the system to initial conditions. This is a characteristic feature of chaotic systems.

In naturally occurring systems, the stably, uniformly oscillating system is very rare but systems showing chaotic oscillation similar to the Chua oscillator occur frequently. A leaf fluttering in the wind, a wave at sea, turbulent eddies in the wake of a ship and the beating of a heart are some examples of chaotic systems.

57.1 Problems

57.1 Sketch the I–V characteristic for the circuit in Figure 57.4 (b) if the $300\,\Omega$ resistors are replaced by $22\,\mathrm{k}\Omega$ resistors and the $1.1\,\mathrm{k}\Omega$ resistor is replaced by a $3.3\,\mathrm{k}\Omega$ resistor.

57.2 A pendulum mass is suspended from the top of a rod which is hinged at the bottom at H and is constrained loosely between two limits at the top as shown in Figure 57.13. The pendulum is driven by a pulsed electromagnetic drive which causes the oscillation amplitude to increase with time. Describe the motion of the pendulum bob as a function of time.

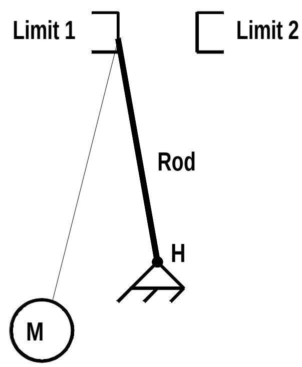

Figure 57.13: Resonantly driven sloppy pendulum.

57.3 Design an electronic circuit which will detect the proximity of the pendulum bob in Problem 57.2 and which will apply a pulse to an electromagnet which is timed so as to increase the amplitude of oscillation of the pendulum.

57.4 A gyrator is an electronic circuit which uses op-amps, resistors and capacitors to simulate an inductor. Figure 57.14 shows the circuit for a Riordan gyrator for which the inductance is given by:

$$Z_{in} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}$$

If $Z_1 = Z_3 = Z_4 = Z_5 = 1\,\text{k}\Omega$ and Z_2 is a capacitor of value $C = 0.01\,\mu$F having an impedance $Z_2 = \frac{1}{j\omega C}$, calculate the inductance simulated by the gyrator.

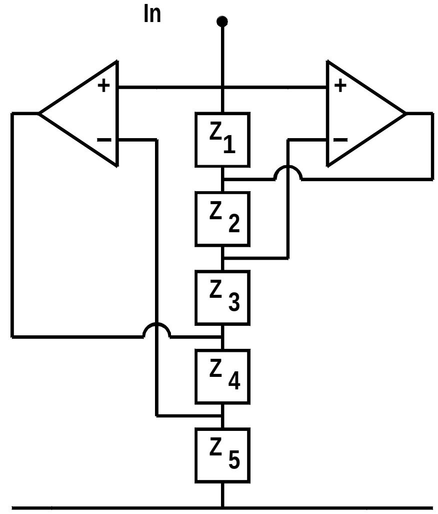

Figure 57.14: Riordan's gyrator simulating an inductance.

57.5 Use the two rules for op-amp operation to derive the relationship:

$$Z_{in} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4}$$

quoted for the gyrator circuit in Problem 57.4.

57.6 Discuss the characteristics of the gyrator, using the circuit in Figure 57.14, which would result from setting $Z_1 = Z_3 = Z_4 = 1\,\text{k}\Omega$, $Z_2 = 0.01\,\mu$F and using a negative resistor circuit for Z_5 of value $Z_5 = -1.5\,\text{k}\Omega$.

57.7 Draw a circuit for a Chua chaotic oscillator which uses a Riordan gyrator in place of the inductor. Calculate suitable values for the components.