Unit 18 Fourier series

- Any repetitive waveform can be synthesized from the sum of sinusoidal waves of appropriate amplitude and phase.
- The frequencies of the Fourier components are the fundamental frequency and integer multiples of this frequency.
- The sharper the corners in the original waveform, the greater will be the amplitudes of the higher frequency Fourier components of the waveform.
- The response of any filter to a repetitive waveform is obtained by summing the responses for each of the Fourier components of the input waveform.

Analysis of waveforms

As this is an introductory course, we will discuss the Fourier analysis of waveforms in graphical terms rather than use the full mathematical treatment which is readily available in any text on Fourier series.

Take a sinusoidal wave of fundamental frequency, f_0, and amplitude 1 as shown in Figure 18.1 (a). Add to this waveform a sinusoid of frequency $3f_0$, the third harmonic, which has an amplitude of 33% of the fundamental. This is shown in Figure 18.1 (b) with the sum of the two waveforms shown in Figure 18.1 (c). Add to this sum a sinusoid of frequency $5f_0$ and amplitude 20% of the fundamental to get Figures 18.1 (d) and (e). Visualize this process continuing for all of the odd harmonics of f_0 given by $f = (2n+1)f_0$ and having amplitudes $\frac{1}{2n+1}$, where n is an integer, eventually leading to the composite synthesized square waveform in Figure 18.1 (f).

As higher harmonics are added in, the corners of the square waveform are sharpened up as shown in Figures 18.1 (a), (c) and (e).

Thus we can see that a square waveform of period T can be considered as the sum of a sinusoidal waveform of fundamental frequency $f_0 = \frac{1}{T}$ combined with sinusoids at the odd harmonics of this fundamental frequency.

In our example we have taken the phase shifts of the harmonics to be zero, that is all of the harmonics are zero at times 0, $\frac{T}{2}$, T, $\frac{3T}{2}, \ldots$ If the

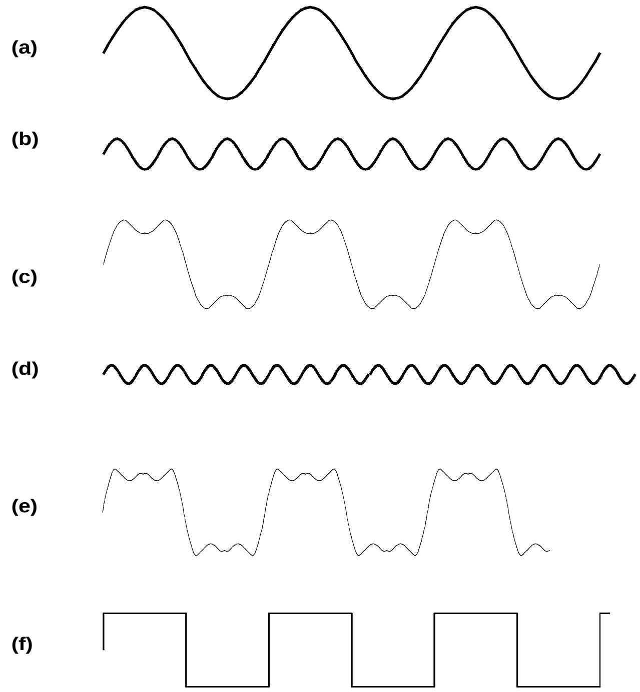

Figure 18.1: Construction of square waveform from Fourier components.

phase of the harmonics is not zero then the waveform synthesized can be quite different even though the amplitudes of the harmonics are unchanged.

Synthesis of filter response

In order to determine the effect of a filter on an arbitrary repetitive waveform, follow the following procedure:

- Obtain the Fourier spectrum of the input waveform.
- Calculate the effect of the filter on each of the Fourier components.
- Combine the modified components to obtain the output waveform.

This procedure can be carried out numerically but often the following graphical method will permit a rapid estimation of the output waveform to be obtained without a long calculation.

Plot the log of the amplitude of each of the Fourier components against the log of the frequency to get a diagram such as that shown in Figure 18.2 (a) which represents the frequency spectrum of the square wave. The small circles indicate the amplitudes of the Fourier component at that frequency.

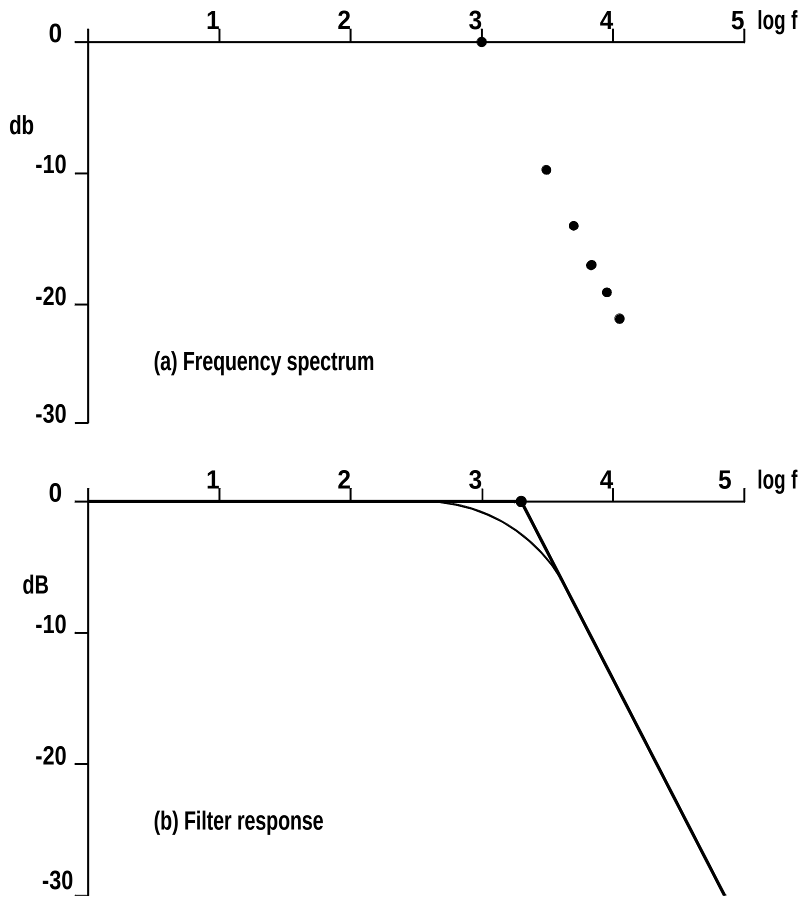

Figure 18.2: (a) Fourier spectrum of square wave (b) Filter response.

If the waveform having this frequency spectrum is passed through a filter with a corner frequency at 2 kHz which has a response curve or Bode plot such as that in Figure 18.2 (b) then the lower frequency components will emerge unchanged but the higher frequencies will be attenuated resulting in the spectrum shown in Figure 18.3.

At each frequency we have multiplied the amplitude of the Fourier component at that frequency by the magnitude of the attenuation of the filter to get the magnitude of the output. This is the powerful feature of the Bode plot approach. Since the log of the amplitude is plotted, all we have to do is, at each frequency, to **add** the logs of the signal and filter responses to get the log of the filter output and thence a log spectrum of the output.

This operation is most easily carried out if the amplitude of the input spectral components and the response curves for the filter(s) are all plotted on a single graph. The spectrum of the output is then obtained by a graphical adding of the signals at each frequency as is shown in Figure 18.3 in which the amplitudes of the Fourier components of the output are indicated by $\times$. For example, at a frequency of 3 kHz or at 3.48 on the log f axis, the filter response is -4 dB and the amplitude of the Fourier component of the square wave is -9 dB which gives the output:

$$-9\,\text{dB} - 4\,\text{dB} = -13\,\text{dB}$$

as indicated by the arrows in Figure 18.3. Remember that the amplitudes are smaller than the reference and therefore the dB values are negative.

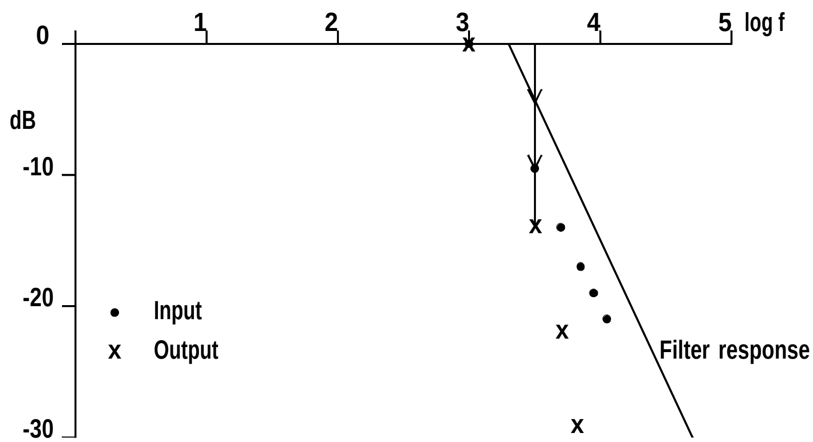

Figure 18.3: Each Fourier component is attenuated by the filter response at that frequency.

This example which we have just discussed represents what happens when a square wave is passed through an RC low pass filter which has a corner frequency which is close to the fundamental frequency of the square wave. The circuit and input and output waveforms are shown in Figure 18.4 and it can be seen that the higher frequency harmonics associated with the sharp edges of the square waveform have been attenuated by the filter to leave a much smoother output waveform.

In our discussion, we have not mentioned the phase shifts which occur in the filter and how they may affect the output waveform. If the filter response is such that the phase delay in the filter is constant over the pass band of the filter then the distortion of the waveform due to phase changes will be minimized.

Also the sharper the corners in the original waveform, the greater will be the amplitudes of the higher frequency components of the waveform.

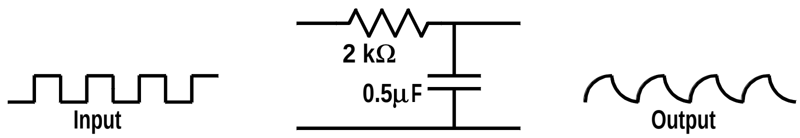

Figure 18.4: Distortion of a square wave by a low pass filter.

18.1 Problems

18.1 Estimate by graphical summation the relative amplitudes of the first two Fourier components of the triangular waveform shown in Figure 18.5.

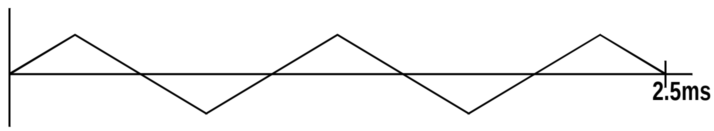

Figure 18.5: Triangular waveform for Problem 18.1.

18.2 Estimate by graphical summation the relative amplitudes and phases of the first four components of the sawtooth waveform shown in Figure 18.6.

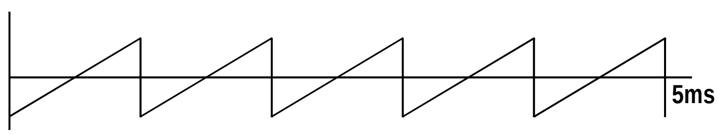

Figure 18.6: Sawtooth waveform for Problem 18.2.

18.3 Sketch a circuit for a CR high pass filter with a corner frequency of 2 kHz. A square wave of fundamental frequency 1 kHz is passed through this filter. Sketch the spectrum of the square wave, the response of the filter and the spectrum of the output waveform. Sketch the shape of the distorted output waveform.

18.4 A square waveform of fundamental frequency 20 kHz is passed through a band pass filter which has a centre frequency of 100 kHz and a 3 dB bandwidth of 15 kHz. Which Fourier components of the square waveform will be passed through the band pass filter? Sketch the output voltage waveform.