Unit 14 Resistances and reactances

- Any combination of resistors, capacitors and inductors, when driven by a sinusoidal signal at an angular frequency ω, can be combined to give a resultant impedance in the form:

$$Z = R + jX$$

where R is the resistance and X is the reactance.

When resistors are connected in series, the resultant resistance is the sum of the individual resistances:

$$R_S = R_1 + R_2 + R_3 + \cdots$$

When resistors, capacitors and inductors are connected in series the resultant impedance is the sum of the individual impedances:

$$Z_S = R_1 + R_2 + \cdots + \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} + \cdots + j\omega L_1 + j\omega L_2 + \cdots$$

This can be reduced, using complex algebra, to give the impedance, Z, as the sum of a resistive component, R, and a reactive component, X.

$$Z_S = R_S + jX_S$$

The units of Z, R and X are ohms.

Similarly, when components are connected in parallel the equivalent of:

$$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots$$

applies with the R_n replaced by the component complex impedance.

Note that the impedance of a capacitor is $\frac{1}{j\omega C}$ and the impedance of an inductor is $j\omega L$. Both of these terms depend on the frequency of the voltage across the component and therefore the impedance changes if the frequency changes. Also the waveform is presumed to be of sinusoidal form and to have been applied for long enough for any start-up transients to have died away and a steady state reached.

14.1 Examples

14.1 Calculate the impedance of a 0.1 μF capacitor connected in series with an 820 Ω resistance at a frequency of 1 kHz.

$$\begin{aligned}
Z &= R + \frac{1}{j\omega C} \\
&= R + \frac{1}{j2\pi f C} \\
&= 820 + \frac{1}{j2\pi 1000 \times 0.1 \times 10^{-6}} \\
&= 820 - \frac{j}{0.000628} \\
&= 820 - j1592 \\
\text{Resistance} \quad R &= 820\,\Omega \\
\text{Reactance} \quad X &= -1592\,\Omega \\
\text{Impedance} \quad Z &= 820\,\Omega - j1592\,\Omega
\end{aligned}$$

Verify the change of sign in the fourth line.

14.2 Calculate the impedance of a 1 MΩ resistor in parallel with a 30 pF capacitor at 40 kHz. (This is the equivalent input impedance for an oscilloscope.)

$$\begin{aligned}
\frac{1}{Z} &= \frac{1}{R} + j\omega C \\
&= \frac{1}{10^6} + j2\pi 40 \times 10^3 \times 30 \times 10^{-12} \\
&= 10^{-6} + j7.54 \times 10^{-6} \\
\text{Therefore} \quad Z &= \frac{1}{10^{-6} + j7.54 \times 10^{-6}} \\
&= \frac{10^{-6} - j7.54 \times 10^{-6}}{(10^{-6})^2 + (7.54 \times 10^{-6})^2} \\
&= \frac{10^{-6} - j7.54 \times 10^{-6}}{5.78 \times 10^{-11}} \\
&= 17\,k\Omega - j130\,k\Omega
\end{aligned}$$

14.2 Problems

14.1 Calculate the impedance of $320\,\Omega$ in series with $10\,\text{mH}$ at $30\,\text{kHz}$.

14.2 Calculate the impedance of $0.2\,\mu\text{F}$, $6\,\text{mH}$ and $680\,\Omega$ all connected in series at a frequency of $4\,\text{kHz}$.

14.3 Calculate the impedance of $2.2\,\text{k}\Omega$ connected in parallel with $0.1\,\mu\text{F}$ at a frequency of $2\,\text{kHz}$.

14.4 Calculate the impedance of a resistance of $12\,\Omega$ connected in parallel with an inductance of $0.2\,\text{H}$ at a frequency of $50\,\text{Hz}$.

14.5 Calculate the impedance of $680\,\Omega$, $0.5\,\mu\text{F}$ and $10\,\text{mH}$ all connected in parallel at a frequency of $7\,\text{kHz}$.

14.6 Convert the complex impedance from the form:

$$Z = R + jX$$

to the form:

$$Z = |Z|\, e^{j\phi}$$

14.7 A sinusoidal voltage waveform is applied across a complex impedance, $Z = R + jX$. Obtain an expression for the average power dissipated in the complex impedance, expressed in terms of the resistance and reactance.