Unit 12 Complex impedance of R, C and L

- The general form of Ohm's law is:

  $$V = Z \times I$$

  where Z is the complex impedance.

  - The impedance of a resistor is R.
  - The impedance of a capacitor is $\frac{1}{j\omega C}$.
  - The impedance of an inductor is $j\omega L$.

- The units of impedance are ohms.

In the last two units we have seen that for a capacitor and for an inductor the current and voltage are out of phase by 90° or $\frac{\pi}{2}$. For a resistor, the voltage and the current are in phase.

The representation of the waveforms by trigonometric functions, such as sin and cos, is cumbersome and a more elegant approach is to use complex numbers, where we use the relationship:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$. (Note that in electronics we use j rather than i to represent $\sqrt{-1}$ because of the possibility of confusion with i when it is used to represent a current.)

A sinusoidally varying voltage can then be represented by the imaginary part of:

$$\begin{aligned} V &= V_0 e^{j\omega t} \\ \text{and then} \quad \frac{dV}{dt} &= V_0 j\omega e^{j\omega t} \\ &= j\omega V \end{aligned}$$

$$\begin{aligned}
\text{So for capacitors} \quad I &= C\frac{dV}{dt} \\
&= j\omega CV \\
\text{giving} \quad V &= \frac{1}{j\omega C} I \\
\text{And for inductances} \quad L\frac{dI}{dt} &= V \\
\text{By integration this becomes} \quad LI &= \int V dt \\
&= \frac{1}{j\omega} V \\
\text{Giving} \quad V &= j\omega LI
\end{aligned}$$

We can now generalize Ohm's law to get:

$$\begin{aligned}
V &= ZI \\
\text{where complex impedance} \quad &= Z \\
\text{and} \quad Z_R &= R \quad \text{for a resistance} \\
Z_C &= \frac{1}{j\omega C} \quad \text{for a capacitor} \\
\text{and} \quad Z_L &= j\omega L \quad \text{for an inductor}
\end{aligned}$$

These three results will be used throughout the text.

12.1 Example

12.1 Calculate the impedance of a 0.1 μF capacitor at a frequency of 19 kHz.

The impedance of a capacitor is:

$$\begin{aligned}
Z_C &= \frac{1}{j\omega C} \\
&= \frac{1}{j2\pi fC} \\
&= \frac{-j}{2\pi fC} \\
&= \frac{-j}{2\pi 19 \times 10^3 \times 0.1 \times 10^{-6}} \\
&= \frac{-j}{0.01193} \\
&= -83.8j\,\Omega
\end{aligned}$$

In this result, the 83.8 gives the numerical relationship between the magnitude or amplitude of the voltage and current waveforms. The

j indicates that the voltage and current sinusoidal waveforms are 90° out of phase with each other and the $-$ sign indicates that the current waveform leads the voltage waveform. (Review Unit 8 and also examine Figure 10.2.)

12.2 Problems

12.1 Calculate the complex impedance of a 100 nF capacitor at frequencies of 1 kHz and 80 kHz.

12.2 Calculate the complex impedance of a 5 mH inductor at 200 Hz and at 100 kHz.

12.3 As the frequency increases, does the magnitude of the impedance of a capacitor increase or decrease?

12.4 As the frequency increases, does the magnitude of the impedance of an inductor increase or decrease?

12.5 A sinusoidal voltage waveform of amplitude 3.1 V and frequency 22.9 kHz is applied across a 0.22 μF capacitor.
Write down the equation for the voltage waveform in complex notation. Calculate the complex impedance of the capacitor. Calculate the equation for the current in the capacitor expressed in complex notation.

12.6 A sinusoidal voltage waveform is applied across a capacitor. Calculate the average power dissipated in the capacitor by integrating the product of the voltage across the capacitor and the current through the capacitor.

12.7 A sinusoidal voltage waveform is applied across an inductor. Calculate the average power dissipated in the inductor by integrating the product of the voltage across the inductor and the current through the inductor.