Unit 16 Quine-McCluskey reduction.

- An implicant corresponds to a group on a Karnaugh map which contains only 1 s.
- A prime implicant corresponds to a group on a Karnaugh map which is not contained wholly within a larger group.
- An essential prime implicant corresponds to a group on a Karnaugh map which contains terms which are not contained in any other group.
- The Quine-McCluskey method is implemented in three stages.
  - Reorder the Canonical terms in increasing order of number of 1 s that appear when the minterm numbers are expressed in binary.
  - Reduce the number of terms by combining terms, starting with the smallest number of 1 s.
  - Select the essential prime implicants, starting with the reduced terms containing the fewest number of literals.

A fundamental concept used in the Quine-McCluskey method of reduction is the concept of a **prime implicant**. A prime implicant is a product term which cannot be combined with other product terms to generate a product term with fewer literals than the original term.

Before examining the concept of a prime implicant it is worth while going back and examining the concept of a prime number. In finding prime numbers such as 5, 11, 29 or 11000001446613353 in the ordinary number system, it is necessary to attempt to factorize a potential prime number. If the number cannot be factorized then it is a prime number. There is no simple rule for proving that a particular number is a prime number apart from trying to factorize it. There is also no simple rule for finding all prime numbers. A computer program can be used to search for factors of a potential prime number but the process takes so much time for large numbers that even the fastest computers can take many years to find all of the primes in a particular range or to factorize a large number. This large required factorization time for "secret" primes is the basis of some of the most secure encryption systems in use today. It is not that the code can not be cracked but that it would probably take to long to crack it.

Similarly, in determining the prime implicants or Boolean prime factors of a Boolean function, there is no easy rule except to try all possible factors or reductions.

On a Karnaugh map (of a 4 input Boolean function) the prime implicants appear as singles, pairs, quads or octets. For instance, the expression

$$\begin{aligned} f(ABCD) &= \overline{A}.\overline{B}.\overline{C}.\overline{D} + \overline{A}.\overline{B}.\overline{C}.D + \overline{A}.B.\overline{C}.\overline{D} + \overline{A}.B.\overline{C}.D + A.\overline{B}.\overline{C}.\overline{D} \\ &\quad + \overline{A}.B.C.D + A.B.\overline{C}.D + A.B.C.\overline{D} + A.B.C.D \\ &= \Sigma m(0,1,4,5,7,8,13,14,15) \end{aligned}$$

This Boolean expression can be represented on the Karnaugh map:

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	1	0	1
$\overline{C}.D$	1	1	1	0
$C.D$	0	1	1	0
$C.\overline{D}$	0	0	1	0

and it is easily seen that that the expression reduces to

$$f(ABCD) = \overline{A}.\overline{C} + B.D + A.B.C + \overline{B}.\overline{C}.\overline{D}$$

which form the prime implicants of the original function.

It can happen that prime implicants can be formed in different ways for a particular set of minterms so we then have a refinement called an **essential prime implicant** which covers at least one minterm which is not covered by any other prime implicant. In the following Karnaugh map we have three prime implicants corresponding to the two squares and the column. However only the left hand square and the column are essential prime implicants as is shown in the second version of the map.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	1	1	1	0
$C.D$	1	1	1	0
$C.\overline{D}$	0	0	1	0

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	1	1	1	0
$C.D$	1	1	1	0
$C.\overline{D}$	0	0	1	0

The Karnaugh map technique shows, visually, that we have obtained the prime implicants but for functions having more than four inputs the Karnaugh map technique fails so it is necessary to use a systematic modification of the Boolean algebra methods in order to obtain the prime implicants.

In the Quine-McCluskey method of reduction, a tabular method is used to generate all of the prime implicants by repeated use of the Boolean identity:

$$X.A + X.\overline{A} = X.(A + \overline{A}) = X.1 = X$$

Successive lists containing don't cares, represented by –, are prepared and a minimal set of prime implicants is identified. This is the tabular equivalent of identifying pairs, quads and octets in the Karnaugh map technique. The procedure is best explained by the use of a worked example.

Minimize a Boolean function for which the minterm expression is:

$$f(x_4, x_3, x_2, x_1, x_0) = \Sigma m(0, 1, 4, 5, 9, 12, 14, 20, 29)$$

Step 1 Write down the tabulation of the minterms.

Minterm	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	0
1	0	0	0	0	1
4	0	0	1	0	0
5	0	0	1	0	1
9	0	1	0	0	1
12	0	1	1	0	0
14	0	1	1	1	0
20	1	0	1	0	0
29	1	1	1	0	1

Step 2. Reorder the rows so that the rows are in a form which makes it convenient to use $A + \overline{A} = 1$ by dividing the rows into groups containing 0, 1, 2, 3 etc 1 s, in each group.

	Minterm	x_4	x_3	x_2	x_1	x_0	
Group 0	0	0	0	0	0	0	$\Leftarrow$
Group 1	1	0	0	0	0	1	$\Leftarrow$
	4	0	0	1	0	0	$\Leftarrow$
Group 2	5	0	0	1	0	1	$\Leftarrow$
	9	0	1	0	0	1	$\Leftarrow$
	12	0	1	1	0	0	$\Leftarrow$
	20	1	0	1	0	0	$\Leftarrow$
Group 3	14	0	1	1	1	0	PI_1
Group 4	29	1	1	1	0	1	PI_2

Step 3 Start with Group 0 and search for rows in the next group which differ in only 1 bit position. Place these pairs of rows in a second list and mark the rows as they are transferred. Place a $-$ in the bit position where the identity $A + \overline{A} = 1$ has been applied. For example, minterm 0 and minterm 1 differ only in the x_0 position so they are entered as 0,1 in the first row in the second list with a $-$ in the position for the x_0 bit. Carry out the same operation on all of the groups. If it is not possible to combine a row with a row in the next group, then that row represents a prime implicant and is marked PI_n. This list that has been formed contains all possible pairs.

Minterm	x_4	x_3	x_2	x_1	x_0	
0,1	0	0	0	0	$-$	$\Leftarrow$
0,4	0	0	$-$	0	0	$\Leftarrow$
1,5	0	0	$-$	0	1	PI_3
1,9	0	$-$	0	0	1	PI_4
4,5	0	0	1	0	$-$	PI_5
4,12	0	$-$	1	0	0	PI_6
4,20	$-$	0	1	0	0	PI_7
12,14	0	1	1	$-$	0	PI_8

Step 4. Repeat this row combining process for the rows marked with $\Leftarrow$ to form a new list corresponding to all possible quads. Again the rows are matched to locate rows which differ in only one bit position. Note that the $-$'s must also match in the pairs of rows. In our example we now get the list:

Minterm	x_4	x_3	x_2	x_1	x_0	
0,1,4,5	0	0	$-$	0	$-$	PI_9
0,4,1,5	0	0	$-$	0	$-$	PI_9

This series of operations has left us with nine prime implicants and the original expression is now reduced to:

$$\begin{aligned} f(x_4,x_3,x_2,x_1,x_0) &= \overline{x_4}.x_3.x_2.x_1.\overline{x_0} + x_4.x_3.x_2.\overline{x_1}.x_0 + \overline{x_4}.\overline{x_3}.\overline{x_1}.x_0 \\ &\quad + \overline{x_4}.\overline{x_2}.\overline{x_1}.x_0 + \overline{x_4}.\overline{x_3}.x_2.\overline{x_1} + \overline{x_4}.x_2.\overline{x_1}.\overline{x_0} \\ &\quad + \overline{x_3}.x_2.\overline{x_1}.\overline{x_0} + \overline{x_4}.x_3.x_2.\overline{x_0} + \overline{x_4}.\overline{x_3}.\overline{x_1} \end{aligned}$$

The next step is to reduce these prime implicants to the set of essential prime implicants.
Step 5 The reduction to the set of essential prime implicants is carried out by forming a prime implicant chart. This is done by tabulating the minterms against the prime implicants as shown below where the minterms included in the prime implicant are indicated by an X.

		0	1	4	5	9	12	14	20	29
	PI_1							X		
→	PI_2									X
	PI_3		X		X					
→	PI_4		X			X				
	PI_5			X	X					
	PI_6			X			X			
→	PI_7			X					X	
	PI_8						X	X		
→	PI_9	X	X	X	X					

If there is only one X in a column then there is only one prime implicant covering that minterm and that prime implicant is an essential prime implicant. The essential prime implicants in the chart above are arrowed.
Step 6 The minterms which are covered by the essential prime implicants can now be removed and we can draw up a new reduced prime implicant chart containing columns for the minterms that were not covered by the first set of essential prime implicants.

	12	14
PI_1		X
PI_6	X	
PI_8	X	X

The minterms in the new table are covered by the prime implicant PI_8 so that the minimum expression for the original expression is now given by the sum of the essential prime implicants as:

$$\begin{aligned}
f(x_4, x_3, x_2, x_1, x_0) &= PI_2 + PI_4 + PI_7 + PI_8 + PI_9 \\
&= (11101) + (0-001) + (-0100) \\
&\quad +(011-0) + (00-0-) \\
&= x_4.x_3.x_2.\overline{x_1}.x_0 + \overline{x_4}.\overline{x_2}.\overline{x_1}.x_0 + \overline{x_3}.x_2.\overline{x_1}.\overline{x_0} \\
&\quad +\overline{x_4}.x_3.x_2.\overline{x_0} + \overline{x_4}.\overline{x_3}.\overline{x_1}
\end{aligned}$$

In this particular example there were no columns containing single terms in the chart resulting from Step 6 so we had to make the best attempt at identifying the essential prime implicants from that chart. If there had been rows containing only one X corresponding to a single minterm then we would have identified that PI_n as an essential prime implicant as in step 5 and then drawn up a new chart as in Step 6.

This particular example contained 5 literals and 9 SoP terms in the original minterm list. The Quine-McCluskey reduction required 5 charts or tables. A system having up to about 20 inputs can be handled on a large sheet of paper but that is the practical limit for the method and larger systems would require the computer based reduction methods which will be discussed in the next few units. The difficulty is that the complexity of the reduction does not increase in proportion to the number of inputs but rather in proportion to some higher power of the number of inputs.

16.1 Problems

16.1 Use the matrix minimization method described in Unit 15 to verify that the Quine-McCluskey worked example does give a minimal expression.

16.2 Use the Quine-McCluskey method to reduce the function specified by:

$$f_1(x_5, x_4, x_3, x_2, x_1, x_0) = \Sigma m(5, 13, 18, 19, 26, 27, 39, 47, 49, 57)$$

16.3 Use the Quine-McCluskey method to reduce the function specified by:

$$\begin{aligned}
f_2(x_5, x_4, x_3, x_2, x_1, x_0) &= \Sigma m(5, 6, 12, 21, 26, 27, 30, 31, \\
&\quad 37, 44, 45, 46, 47, 53, 58, 62)
\end{aligned}$$