Unit 14 Karnaugh maps.

- A **prime implicant** is a product term which cannot be further simplified by combination with other terms.
- The Gray code ordering of Karnaugh map rows and columns implies that the canonical forms of the sums of products in any adjacent row or column differ only in one bit position.
- The Boolean reduction

$$F = A.B + A.\overline{B} = A.(B + \overline{B}) = A.1 = A$$

can then be repeatedly applied to reduce groups to prime implicants and give a minimized expression.

The Karnaugh map technique gives a systematic tabular method for reducing a large Boolean expression. The method is fast and also avoids the, sometimes, intricate manipulations associated with the reduction of expressions using Boolean algebra.

There are essentially seven steps in the procedure.

Step 1. Start with the truth table which specifies the required system. We initially consider the trivial example shown below.

A	B	Q
0	0	0
0	1	0
1	0	1
1	1	1

Step 2. Examine this truth table and determine the rows for which the Boolean product terms are $Q = 1$. In this case we have

Row $A = 1$ and $B = 0$ for which $A.\overline{B} = 1$
Row $A = 1$ and $B = 1$ for which $A.B = 1$

Step 3. We now prepare a grid with the variables along the two axes. This is the Karnaugh map. For the two variable case it is trivial and is shown below. For the case where there are more than two variables, the axes will be marked out using the Gray code number system so that only one bit changes in going from one row to the next row or from one column to the next column.

	$\overline{B}$	B
$\overline{A}$		
A		

Step 4. On this Karnaugh map we mark in the input combinations for which the output $Q = 1$.
Step 5. For the combinations for which the output $Q = 0$ we mark in a 0 at the location on the Karnaugh map.

	$\overline{B}$	B
$\overline{A}$	0	0
A	1	1

Step 6. Draw an ellipse around the adjacent pair of 1 s on the Karnaugh map as shown.

Step 7. This pair of 1 s which have been high lighted form a Boolean expression which can be reduced as follows:

$$Q = A.\overline{B} + A.B = A.(\overline{B} + B) = A.1 = A$$

Thus adjacent rows or columns on a Karnaugh map contain terms which differ in only one variable and that variable can be removed from the pair by use of $\overline{A} + A = 1$. This graphical reduction is possible because we use Gray coding in assigning the axes of the Karnaugh map. The great advantage of the Karnaugh map procedure is that it enables us very quickly to recognize those entries in the map which differ in only one variable.

Now let us examine some Karnaugh maps constructed from four inputs A, B, C and D. In the following Karnaugh map we see that there are two pairs of 1 s which have been circled and which allow the simplification of the Boolean expression

$$Q = \overline{A}.\overline{B}.\overline{C}.\overline{D} + \overline{A}.\overline{B}.\overline{C}.D + A.\overline{B}.C.\overline{D} + A.B.C.\overline{D}$$

into the **prime implicants**, or terms which cannot be further simplified, to give

$$Q = \overline{A}.\overline{B}.\overline{C} + A.C.\overline{D}$$

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	0	0	0
$\overline{C}.D$	1	0	0	0
$C.D$	0	0	0	0
$C.\overline{D}$	0	0	1	1

Because of the wraparound property of the Gray code number system, the codes for entries on opposite sides of the Karnaugh map, if they are on the same row or column, will differ in only one bit and therefore will form pairs as shown in the Karnaugh map below where these edge pairs have been identified by half ellipses.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	0	0	0	0
$C.D$	1	0	0	1
$C.\overline{D}$	0	0	1	0

This Karnaugh map then gives a reduced Boolean expression

$$Q = \overline{B}.C.D + A.B.\overline{D}$$

The Karnaugh map also allows the identification of quads or groups of four inputs as shown in the two Karnaugh maps below.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	0	0
$\overline{C}.D$	0	0	0	0
$C.D$	1	1	1	1
$C.\overline{D}$	0	0	0	0

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	1
$\overline{C}.D$	0	0	1	1
$C.D$	0	0	0	0
$C.\overline{D}$	0	0	0	0

Groups of eight or octets can also be identified and combined.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	0	0
$\overline{C}.D$	1	1	1	1
$C.D$	1	1	1	1
$C.\overline{D}$	0	0	0	0

When the groups on the Karnaugh map overlap, as shown on the map below, there are two legitimate forms of the Boolean expression. One of the groups can be simplified and the single element can remain by itself to give

$$Q = \overline{C}.D + \overline{A}.B.C.D$$

or both groups can be simplified to give a Boolean expression which has overlapping terms

$$Q = \overline{C}.D + \overline{A}.B.D$$

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	0	0
$\overline{C}.D$	1	1	1	1
$C.D$	0	1	0	0
$C.\overline{D}$	0	0	0	0

In the specification of the requirements for a Boolean system, it frequently occurs that the output is independent of certain combinations of input conditions because it is not possible for these conditions or input combinations to arise. These particular conditions are then referred to as "don't care" states. For example, considering transport mode combinations, the groups (man on foot), (man on horse) and (man on bicycle) would be considered but the combination (horse on bicycle) would usually be regarded as a "don't care" combination as it would not normally arise.

When systems which contain "don't care" states are specified the usual form is:

$$f(ABCD) = \Sigma m(0,\ 1,\ 4,\ 6,\ 8) + d(3,\ 12,\ 14)$$

where states 3, 12 and 14 are the "don't care" states. When a Karnaugh map reduction is being carried out on such a system we have the option of including or excluding these "don't care" states in the groups depending on which option gives the greatest simplification.

The three Karnaugh maps shown below contain the same information about the system but the simplifications due to the different groupings are different when the options of including or excluding the "don't care" terms are exploited. We have represented the "don't care" terms in the Karnaugh map by X s instead of the 0 s and 1 s used normally.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	1	X	1
$\overline{C}.D$	1	0	0	0
$C.D$	X	0	0	0
$C.\overline{D}$	0	1	X	0

Which gives the Boolean expression:

$$Q = \overline{A}.\overline{B}.\overline{C} + \overline{A}.\overline{C}.\overline{D} + A.\overline{B}.\overline{C}.\overline{D} + \overline{A}.B.C.\overline{D}$$

On the other hand, it is possible to use the "don't care" states as if they were logic 1s and then this Karnaugh map is obtained:

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	1	X	1
$\overline{C}.D$	1	0	0	0
$C.D$	X	0	0	0
$C.\overline{D}$	0	1	X	0

This Karnaugh map, which includes the "don't care" options within groups, gives an expression which contains fewer literals and also fewer terms.

$$Q = \overline{C}.\overline{D} + \overline{A}.\overline{B}.\overline{C} + B.C.\overline{D}$$

However, inspection of this last Karnaugh map reveals that it is possible to create a group of four which wraps from top to bottom as shown in the map below.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	1	X	1
$\overline{C}.D$	1	0	0	0
$C.D$	X	0	0	0
$C.\overline{D}$	0	1	X	0

This minimized Karnaugh map gives the minimized function:

$$Q = \overline{C}.\overline{D} + B.\overline{D} + \overline{A}.\overline{B}.\overline{C}$$

A template for preparing a Karnaugh map from the minterm list is obtained by writing the minterm number into the Karnaugh map at the appropriate position for that minterm. The template for a four input system is shown below.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	4	12	8
$\overline{C}.D$	1	5	13	9
$C.D$	3	7	15	11
$C.\overline{D}$	2	6	14	10

This template is used by taking the terms or numbers in the minterm list and writing a 1 in the appropriate position in the template and writing 0 s elsewhere.

14.1 Examples

14.1 Obtain a simplified Boolean expression, containing prime implicants, from the Karnaugh map shown below.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	1	1	1	1
$C.D$	0	1	0	0
$C.\overline{D}$	0	0	1	0

The elements of this map can be grouped in two ways as follows:

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	1	1	1	1
$C.D$	0	1	0	0
$C.\overline{D}$	0	0	1	0

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	0	0	1	0
$\overline{C}.D$	1	1	1	1
$C.D$	0	1	0	0
$C.\overline{D}$	0	0	1	0

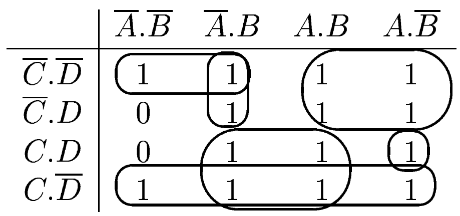
This gives the alternative forms of the Boolean expression:

$$\begin{aligned} Q &= \overline{C}.D + \overline{A}.B.D + A.B.\overline{D} \\ &= \overline{C}.D + \overline{A}.B.C.D + A.B.\overline{D} \end{aligned}$$

14.2 Simplify the Karnaugh map given below.

	$\overline{A}.\overline{B}$	$\overline{A}.B$	$A.B$	$A.\overline{B}$
$\overline{C}.\overline{D}$	1	1	1	1
$\overline{C}.D$	0	1	1	1
$C.D$	0	1	1	1
$C.\overline{D}$	1	1	1	1

We could draw the ellipses on the map as shown below



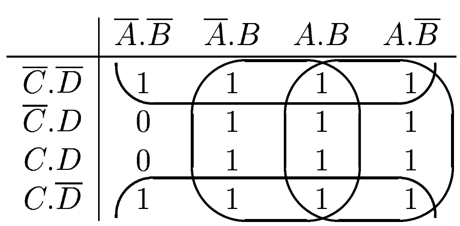
This very inefficient set of groups actually represents the expression which formed the starting point of Problem 13.5 in Unit 13.

$$Q = C.B + A.\overline{B}.C.D + C.\overline{D} + A.\overline{C} + \overline{A}.B.\overline{C} + \overline{B}.\overline{C}.\overline{D}$$

However a more reasonable assignment of the groups would be

$$Q = A + B + \overline{D}$$

which is represented by the Karnaugh map below which contains three octets. As a measure of the power of the Karnaugh map technique, you should compare the complexity of the Boolean algebra simplification which was necessary for Problem 13.5 to the simplicity of the graphical allocation of the groups in this example.



14.2 Problems

14.1 The truth table for a particular system is shown below.
Obtain the minterm list and write out the canonical Boolean expression for the system.
Use Boolean algebra to simplify the expression.
Draw the Karnaugh map of the system and use it to obtain the minimum Boolean expression.
Verify that the Boolean algebra and the Karnaugh map methods of simplification give the same result.

A	B	C	Q
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

14.2 The minterm list for a particular system is

$$f(ABCD) = \Sigma m(1,\ 2,\ 3,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 14)$$

Draw up the truth table for the system.
Draw the Karnaugh map and use it to obtain a minimum Boolean expression. Use the template on page 87
Indicate which terms of the Boolean expression correspond to each of the groups.
Draw the logic circuit diagram for the system.

14.3 Two binary bits, A and B, and a carry-in, C_{in}, are to be added to give a sum, S, and a carry-out, C_{out}. Construct the truth table for the three inputs and the two outputs from the system. From the truth table, obtain the Boolean expression for S and C_{out}. Can this expression be minimized further by use of the Karnaugh map method?

14.4 The symbol, $\oplus$, is used to represent the Exclusive-OR operator in Boolean algebra. Construct the Karnaugh map for the function:

$$Q = A \oplus B \oplus C \oplus D$$