Unit 13 Boolean minimization.

- The terms of the canonical form which represents a particular truth table can be manipulated using the Boolean algebra so as to obtain a circuit having:
  - a minimum depth (two level minimization)
  - or a minimum no of gates
  - or a minimum no of interconnections
- The possibility of proving that a particular form of a reduced expression is truly a minimum is considered.

The canonical form of the Boolean algebra expression which defines a truth table having n inputs or literals will contain up to 2^n terms which specify the rows of the truth table for which $Q = 1$. Even though the minterm or maxterm list formulation gives the same information in a very compact description of the truth table, there is still the problem of implementing the truth table in a logic gate circuit.

A truth table for a simple system is shown below and the AND-OR gate circuit which implements this truth table is shown in Figure 13.1.

A	B	C	Q
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

If there are m out of a total of 2^n truth table rows for which $Q = 1$, we will require m n-input AND gates to implement the truth table as an electronic circuit and a single m-input OR gate to form the Sum of Products. This is a trivial problem for three inputs but it can be very inconvenient if there

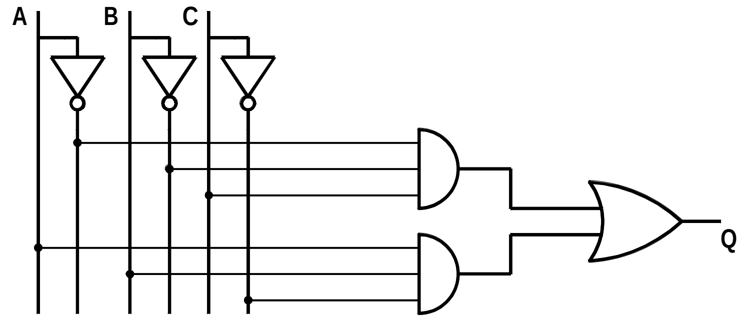

Figure 13.1: Truth table for $Q = \overline{A}\overline{B}C + AB\overline{C}$ and the Sum of Products circuit.

are more than eight inputs as the chip count and the track count increase dramatically, pushing up the cost and power consumption and reducing the circuit speed.

We would therefore wish to have techniques which will reduce the circuit complexity and give a minimum chip count, minimum track complexity, minimum power consumption and a maximum speed circuit. Not all of these targets can be achieved simultaneously but the trend is in the same direction for all of the constraints so we need not make too significant a distinction between the different targets and we will simply search for a minimum circuit for an arbitrary truth table.

A number of techniques have been developed for circuit minimization and we will discuss them in this and the next five Units. The minimization techniques are all ultimately based on the use of Boolean algebra and use the postulates and theorems which are summarized at the start of Unit 9 on Boolean algebra. The Boolean algebraic simplification techniques are best explained by taking a number of examples.

13.1 Examples

13.1 Simplify the following canonical expression to a Sum of Products.

$$Q = A.B.C + A.B.\overline{C} + \overline{A}.B.C$$

This simplification uses $A + \overline{A} = 1$.

$$\begin{aligned} Q &= A.B.C + A.B.\overline{C} + \overline{A}.B.C \\ &= A.B.(C + \overline{C}) + \overline{A}.B.C \\ &= A.B.1 + \overline{A}.B.C \\ &= A.B + \overline{A}.B.C \end{aligned}$$

Which is a simplified expression in the form of Sum of Products, as required.

However, consider the following alternative simplification:

$$\begin{aligned} Q &= A.B.C + A.B.\overline{C} + \overline{A}.B.C \\ &= A.B.C + \overline{A}.B.C + A.B.\overline{C} \\ &= (A + \overline{A}).B.C + A.B.\overline{C} \\ &= B.C + A.B.\overline{C} \end{aligned}$$

This expression is as simple as the previous expression but is different from the previous expression. So we must bear in mind that there may be a number of possible simplifications of canonical Boolean expressions which are equally simple, as in this case. When there are more canonical terms in the expression and also more literals, we must consider the possibility that we may not have found the simplest of the possible expressions (assuming that we have some definition of "simplest" expression).

For instance, if we use the Boolean expansion $X = X + X$ we get:

$$\begin{aligned} Q &= A.B.C + A.B.\overline{C} + \overline{A}.B.C \\ &= A.B.C + \overline{A}.B.C + A.B.C + A.B.\overline{C} \\ &= (A + \overline{A}).B.C + A.B.(C + \overline{C} \\ &= B.C + A.B \end{aligned}$$

which is a simpler expression than either of the other two!

13.2 Convert the following expression into the minterm list form. Simplify the expression, using Boolean algebra, and draw the circuit.

$$\begin{aligned} Q = \; & \overline{A}.\overline{B}.\overline{C}.\overline{D} + \overline{A}.\overline{B}.C.D + \overline{A}.B.C.D + A.\overline{B}.\overline{C}.\overline{D} \\ & + A.\overline{B}.C.\overline{D} + A.\overline{B}.C.D + A.B.C.\overline{D} + A.B.C.D \end{aligned}$$

One way to obtain the minterm list is to convert the literals to binary while taking care to maintain the sequence and then to convert the binary number to decimal and obtain the minterm list directly.

$$\begin{aligned} Q &= \Sigma m(0000,\ 0011,\ 0111,\ 1000,\ 1010,\ 1011,\ 1110,\ 1111) \\ &= \Sigma m(0,3,7,8,10,11,14,15) \end{aligned}$$

A Boolean simplification could proceed as follows:

$$\begin{aligned} Q &= \overline{A}.\overline{B}.\overline{C}.\overline{D} + \overline{A}.\overline{B}.C.D + \overline{A}.B.C.D + A.\overline{B}.\overline{C}.\overline{D} \\ &\quad + A.\overline{B}.C.\overline{D} + A.\overline{B}.C.D + A.B.C.\overline{D} + A.B.C.D \\ &= (\overline{A} + A).\overline{B}.\overline{C}.\overline{D} + \overline{A}.(\overline{B} + B).C.D \\ &\quad + A.\overline{B}.C(\overline{D} + D) + A.B.C.(\overline{D} + D) \\ &= \overline{B}.\overline{C}.\overline{D} + \overline{A}.C.D + A.\overline{B}.C + A.B.C \\ &= \overline{B}.\overline{C}.\overline{D} + \overline{A}.C.D + A.C.(\overline{B} + B) \\ &= \overline{B}.\overline{C}.\overline{D} + \overline{A}.C.D + A.C \\ &= \overline{B}.\overline{C}.\overline{D} + C(\overline{A}.D + A) \\ &= \overline{B}.\overline{C}.\overline{D} + C(D + A) \text{ by use of T4} \\ &= \overline{B}.\overline{C}.\overline{D} + C.D + A.C \end{aligned}$$

This can then be implemented as a minimum circuit as shown in Figure 13.2. Note that in this circuit diagram the bus tracks for $\overline{A}$ and B have no connections and are not strictly necessary but have been left in place as a standard feature.

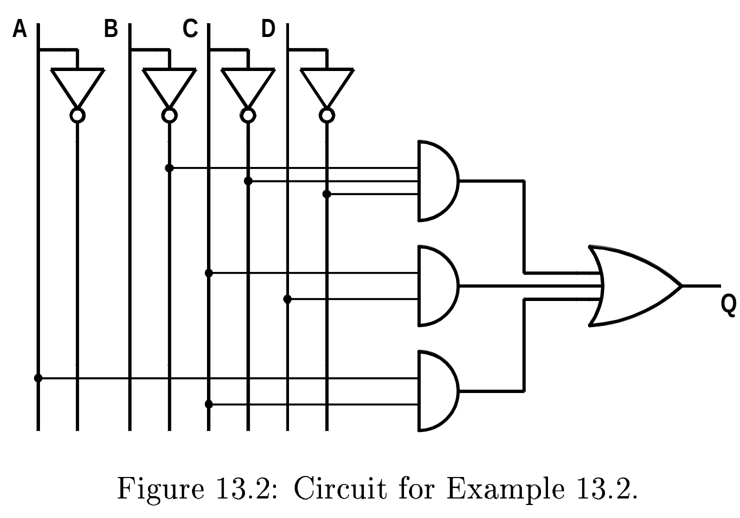

Figure 13.2: Circuit for Example 13.2.

13.2 Problems

13.1 Simplify the following expression

$$F1 = AB\overline{C} + A\overline{BC} + \overline{AB}C + \overline{A}BC$$

13.2 Simplify the following expression

$$F2 = AB\overline{C} + \overline{A}BC + ABC$$

13.3 Simplify the following expression

$$F3 = \overline{A}B\overline{C} + AB\overline{C} + \overline{A}BC + A\overline{B}C$$

13.4 Simplify the following expression

$$\begin{aligned} F4 &= \overline{ABCD} + \overline{A}B\overline{CD} + \overline{ABC}D + \overline{A}B\overline{C}D \\ &\quad + AB\overline{C}D + \overline{A}BCD + ABCD + A\overline{B}C\overline{D} \end{aligned}$$

13.5 Show, by use of the postulates and theorems of Boolean algebra and by direct calculation using a truth table, that the following identity is true.

$$C.B + A.\overline{B}.C.D + C.\overline{D} + A.\overline{C} + \overline{A}.B.\overline{C} + \overline{B}.\overline{C}.\overline{D} = A + B + \overline{D}$$

Draw the logic gate diagram which represents each side of the identity.