Unit 10 DeMorgan's Theorem.

The two general complementary forms of DeMorgan's Theorem are:

$$\begin{aligned}\overline{(A+B+C+\ldots)} &= \overline{A}.\overline{B}.\overline{C}\ldots \\ \overline{A}+\overline{B}+\overline{C}+\ldots &= \overline{(A.B.C\ldots)}\end{aligned}$$

There is one particular theorem of Boolean algebra that is so useful that we will discuss it in a unit by itself. The theorem is due to the English mathematician and logician, Augustus DeMorgan who was professor at University College, London between 1828 and 1866. Some of his work predated the more formal statements of what are now called Boolean algebra but DeMorgan, nevertheless, made many valuable contributions which were later incorporated into the Boolean formalism. One of the books by which he was best known, *Budget of Paradoxes*, is an exposé of scientific and mathematical cranks which was only published after his death.

DeMorgan's Theorem allows a function to be complemented and therefore appears in two forms which are complements of each other. In the formalism of Boolean algebra the theorem can be stated as:

$$\begin{aligned}\text{Complement of Sums} &= \text{Product of Complements} \\ \overline{(A+B)} &= \overline{A}.\overline{B} \\ \text{Sum of Complements} &= \text{Complement of Products} \\ \overline{A}+\overline{B} &= \overline{(A.B)}\end{aligned}$$

These two forms of DeMorgan's Theorem can be proved from the postulates of Boolean and theorems algebra which were discussed in Unit 9. The references in the following proofs are to the postulate and theorem numbers in Unit 9.

$$\begin{aligned}\text{Complement of Sums} &= \text{Product of Complements} \\ \overline{(A+B)} &= \overline{A}.\overline{B}\end{aligned}$$

We prove this first form of DeMorgan's theorem by using postulate 4 which defines complements, $A+\overline{A}=1$ and $A.\overline{A}=0$.

So we take as a starting point:

$$\begin{aligned}
A+B+\overline{A}.\overline{B} &= (A+B+\overline{A}).(A+B+\overline{B}) && \text{by P2}\\
&= (B+1).(A+1) && \text{by P4}\\
&= 1.1 && \text{by T2}\\
&= 1 &&\\
\text{and also } (A+B).\overline{A}.\overline{B} &= \overline{A}.\overline{B}.(A+B) && \text{by P1}\\
&= \overline{A}.\overline{B}.A+\overline{A}.\overline{B}.B && \text{by P2}\\
&= \overline{B}.\overline{A}.A+\overline{A}.\overline{B}.B &&\\
&= \overline{B}.0+\overline{A}.0 &&\\
&= 0 &&
\end{aligned}$$

So we have shown that $\overline{\overline{A+B}} = A+B$ and $\overline{A}.\overline{B}$ together satisfy equations of the form $A+\overline{A}=1$ and $A.\overline{A}=0$ and are complements and therefore we obtain the result that $\overline{A+B} = \overline{A}.\overline{B}$ as required.

The second form of DeMorgan's Theorem is:

$$\begin{aligned}
\text{Complement of Sums} &= \text{Product of Complements}\\
\overline{A}+\overline{B} &= \overline{(A.B)}
\end{aligned}$$

This second form of DeMorgan's Theorem is proved in a complementary way from the postulates by starting with:

$$\begin{aligned}
\overline{A}+\overline{B}+A.B &= (\overline{A}+\overline{B}+A).(\overline{A}+\overline{B}+B) && \text{by P2}\\
&= (\overline{B}+1).(\overline{A}+1) && \text{by T2}\\
&= 1 &&\\
\text{and } (\overline{A}+\overline{B}).A.B &= \overline{A}.A.B+\overline{B}.A.B && \text{by P2}\\
&= B.0+A.0 && \text{by P4}\\
&= 0 &&
\end{aligned}$$

So that $\overline{A}+\overline{B}$ and $A.B$ also satisfy the condition for the existence of complements, P4, and therefore we obtain $\overline{A.B} = \overline{A}+\overline{B}$.

However, most people's memory for algebraic formulae can be erratic and an alternative formulation of DeMorgan's theorem is shown in Figure 10.1.

This formulation essentially allows the interchange of NAND and NOR logic gates by the use of inverters within circuits and this application is the principal use of DeMorgan's theorem in circuit analysis and design.

To continue the approach to illustrative proofs taken in Unit 9, we present an enumerative truth table verification of the two forms of the theorem:

A	B	$A+B$	$\overline{A+B}$	$\overline{A}$	$\overline{B}$	$\overline{A}.\overline{B}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Table 10.1. Tabular proof of $\overline{(A+B)} = \overline{A}.\overline{B}$.

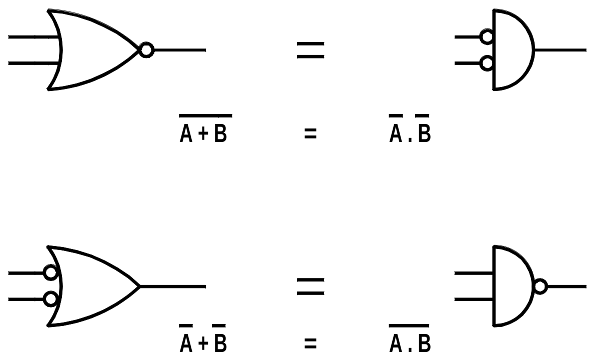

Figure 10.1: Circuit statement of DeMorgan's theorem.

A	B	$\overline{A}$	$\overline{B}$	$\overline{A}+\overline{B}$	$A.B$	$\overline{A.B}$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

Table 10.2. Tabular proof of $\overline{A}+\overline{B}=\overline{A.B}$.

By inspection, it can be seen that the columns corresponding to $\overline{(A+B)}$ and $\overline{A}.\overline{B}$ in the first table are the same and that the columns corresponding to $\overline{A}+\overline{B}$ and $\overline{A.B}$ in the second table are also the same.

There are also two slightly less serious, but nevertheless valid, illustrations of DeMorgan's Theorem. First, represent a working airplane by A, a working parachute by P and the state of being dead by D.

Then $\overline{D}=A+P$ is the equation which represents the state of either being in a working airplane or hanging from a working parachute.

And $D=\overline{A+P}$ represents the state of having neither a working airplane nor a working parachute.

But we can also represent the state of having a non working airplane and a non working parachute by $D=\overline{A}.\overline{P}$.

So we see that $\overline{A+P}=\overline{A}.\overline{P}$.

An alternative illustration is to assign S to smoking a cigarette, C to having a cigarette in the packet and M to having matches in the matchbox.

We then see that $S=C.M$ means that you can have a smoke if you have a cigarette and a match.

And $\overline{S}=\overline{C}.\overline{M}$ represents not smoking because you do not have a cigarette and a match.

However you can interpret $\overline{S}=\overline{C}+\overline{M}$ as having no cigarette or having no matches which again gives the relationship $\overline{C}+\overline{M}=\overline{C.M}$.

While these two examples do not fit neatly into mainstream digital electronics, they do illustrate a very important aspect of digital electronics. Digital electronics is used to carry out logical operations on signals or symbols which represent quantities in the real world and it is very important to be able to assign well defined symbols to real world variables so that computations can be carried out and an answer or result obtained.

10.1 References

DeMorgan, Augustus (1872) *Budget of Paradoxes*

10.2 Problems

10.1 Show that these two C programming language expressions are equivalent.

```
!(number1 == 0.0 && number2 == 0.0)
  number1 != 0.0 || number2 != 0.0
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10.2 Construct this circuit using only NOR gates. Derive the Boolean expression for the output, Q.

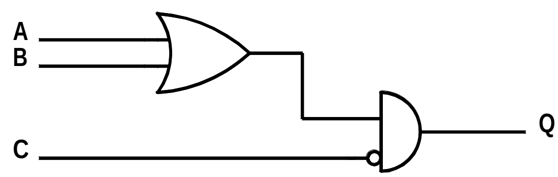

Figure 10.2: Circuit for problem 10.2.

10.3 Construct this circuit using only NAND gates. Derive the Boolean expression for the output, Q.

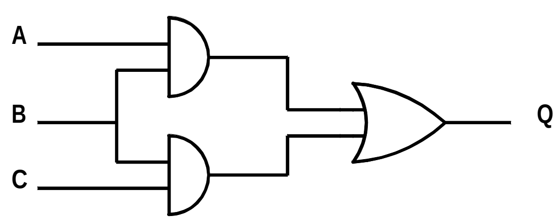

Figure 10.3: Circuit for problem 10.3.