Unit 9 Boolean Algebra.

The basic postulates of Boolean algebra are:

P1.	Commutation operations	$A + B$	$=$	$B + A$
		$A.B$	$=$	$B.A$
P2.	Distribution operations	$A + (B.C)$	$=$	$(A + B).(A + C)$
		$A.(B + C)$	$=$	$A.B + A.C$
P3.	Identity elements exist	$A + 0$	$=$	A
		$A.1$	$=$	A
P4.	Complements exist	$A + \overline{A}$	$=$	1
		$A.\overline{A}$	$=$	0

and some useful theorems are:

T1	$A + A$	$=$	A	$A.A$	$=$	A
T2	$A + 1$	$=$	1	$A.0$	$=$	0
T3	$A + A.B$	$=$	A	$A.(A + B)$	$=$	A
T4	$A + \overline{A}.B$	$=$	$A + B$	$A.(\overline{A} + B)$	$=$	$A.B$

Boolean algebra is generally taken to be that branch of mathematics which treats the operation of the operators OR (+), AND (.) and INVERT or COMPLEMENT ($\overline{A}$) on binary logic signals. The subject is called after George Boole who was Professor of Mathematics at what is now University College Cork and who played a major part in the development of the subject.

Boolean algebra can be treated as a purely mathematical subject with its own strict formalism but, since we use Boolean algebra in digital electronics, we will present parallel mathematical, truth table and electronic gate treatments.

One warning must be given; the symbols + and . as used in Boolean algebra denote logical OR and logical AND. They do not have the same meaning as the plus and multiplication signs as used in arithmetic (binary and decimal) and some results are not as you would expect. For example in Boolean algebra, $1 + 1 = 1$ whereas in decimal arithmetic $1 + 1 = 2$ and in single bit binary arithmetic, $1 + 1 = 0$ with a carry to the next most significant bit, if used.

The operations of Boolean algebra are based on the following postulates:

P1.	Commutation operations	$A + B$	$=$	$B + A$
		$A.B$	$=$	$B.A$
P2.	Distribution operations	$A + (B.C)$	$=$	$(A + B).(A + C)$
		$A.(B + C)$	$=$	$A.B + A.C$
P3.	Identity elements exist	$A + 0$	$=$	A
		$A.1$	$=$	A
P4.	Complements exist	$A + \overline{A}$	$=$	1
		$A.\overline{A}$	$=$	0

While these are the postulates of Boolean algebra and form the starting point for the subject, the reasonableness of these postulates can be illustrated by enumeration. For instance if we take postulate P4, $A+\overline{A} = 1$, the possible values of A are 0 or 1 so we can prepare a truth table which enumerates all possibilities and demonstrates the postulate.

A	$\overline{A}$	$A + \overline{A}$
0	1	1
1	0	1

Similarly, the other postulates can be illustrated by enumeration, using a truth table (see Problems).

Figure 9.1 shows the first form of Postulate 2 in the form of a logic gate.

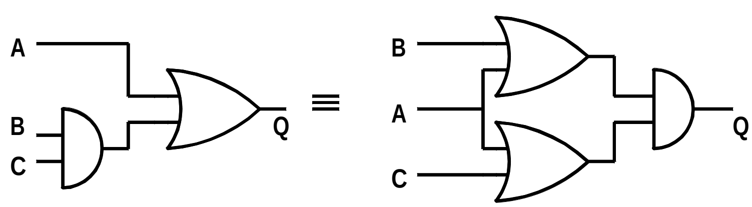

Figure 9.1: Logic circuit for Postulate 2; $A + (B.C) = (A + B).(A + C)$.

When we use a truth table to verify this postulate, we obtain a result as shown in the table.

A	B	C	$B.C$	$A+B.C$	$A+B$	$A+C$	$(A+B).(A+C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Using these postulates, it is then possible to prove a number of theorems which are of use in simplifying and manipulating the Boolean expressions encountered in digital electronics. In tabular form the theorems are:

T1	$A+A = A$	$A.A = A$
T2	$A+1 = 1$	$A.0 = 0$
T3	$A+A.B = A$	$A.(A+B) = A$
T4	$A+\overline{A}.B = A+B$	$A.(\overline{A}+B) = A.B$

Let us now prove some of these theorems from the postulates. Take the first relationship, the idempotent relationship, $A+A=A$. We have:

$$\begin{aligned} A+A &= (A+A).1 && \text{by P3.} \\ &= (A+A).(A+\overline{A}) && \text{by P4.} \\ &= A+A.\overline{A} && \text{by P2a reversed.} \\ &= A+0 && \text{by P4.} \\ &= A && \text{by P3} \end{aligned}$$

For the proof of the second form of T3, $A.(A+B)=A$ which is also called the absorption Theorem, we have:

$$\begin{aligned} A.(A+B) &= (A+0).(A+B) && \text{by P3.} \\ &= A+0.B && \text{by P2.} \\ &= A+0 && \text{by T2.} \\ &= A && \text{by P3.} \end{aligned}$$

This same relationship can also be proved using the enumeration approach with the aid of a truth table:

A	B	$(A+B)$	$A.(A+B)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

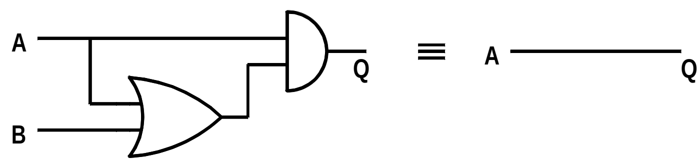

Figure 9.2: Logic gate circuit representing Theorem 3b; $A.(A+B)=A$.

And finally, this same Theorem can be illustrated in the form of two equivalent digital logic circuit diagrams as shown in Figure 9.2.

The reader is invited to attempt to establish the truth of the remainder of the Theorems by using each of the three approaches.

9.1 References

Barry, P.D. (ed) (1969), *George Boole*, Cork University Press

MacHale, D. (1985) *George Boole, His Life and Works,* Boole Press Dublin.

Floyd, T. L. (1994), *Digital Fundamentals, 5th ed.*, Merrill Maxwell.

Lala, P.K. (1996), *Practical Digital Logic Design and Testing*, Prentice Hall.

9.2 Problems

9.1 Draw the logic gate circuits which represent each side of the equation for the second form of Postulate 2, that is $A.(B+C)=A.B+A.C$.

9.2 Prepare a truth table, containing rows for all possible combinations of A, B and C, and show that the explicit calculations of $A.(B+C)$ and $A.B+A.C$ are equivalent.

9.3 Draw the logic gate circuit which implements $A.(B+C)$ and $A.B+A.C$.

9.4 Prove the two results of theorem T4, $A+\overline{A}.B=A+B$ and $A.(\overline{A}+B)=A.B$ from the postulates P1 to P4.

9.5 Explain the context in which each of the following statements is valid.

(a) 1+1 = 0

(b) 1+ 1 = 1

(c) 1+1 = 2

(d) 1+1 = 10

9.6 Construct the truth table for the circuit shown in Figure 9.3 and write down the Boolean expression for the circuit.

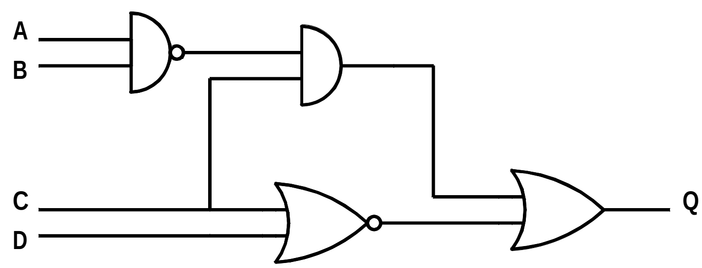

Figure 9.3: Logic gate circuit for Problem 9.6.

9.7 Construct the truth table for the circuit shown in Figure 9.4 and write down the Boolean expression for the circuit.

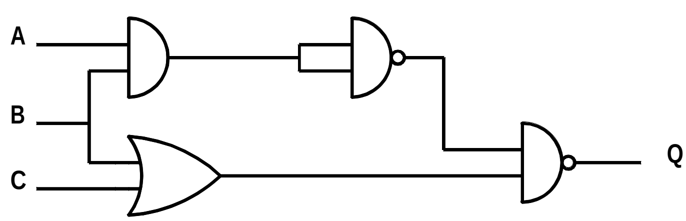

Figure 9.4: Logic gate circuit for Problem 9.7.